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Edited by T. W. CHAUNDY, U. S. HASLAM-JONES, J. H. C. THOMPSON


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# INEQUALITIES FOR INTEGRAL FUNCTIONS

*By* J. CLUNIE (*Imperial College, London*)

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**1.** Let $\chi(r)$ be a real function with a continuous $n$th derivative for all large $r$. Let $S$ be the class of integral functions $f(z)$ satisfying

$$M(r,f) \equiv \max|f(z)| \leqslant \chi(r) \quad (|z| = r \geqslant r_0)$$

with $r_0$ fixed. Define

$$\lambda_n(r) = \text{u.b.}\, M_n(r,f) \quad (f \in S),$$

where $$M_n(r,f) = \max|f^{(n)}(z)| \quad (|z| = r).$$

Then, given $\kappa > 1$, there is an infinite sequence of $r$ such that

$$\lambda_n(r) < n!\, \kappa(e/n)^n \chi^{(n)}(r). \tag{1}$$

This result was proved by Hayman (**2**) for $n = 1, 2$, and in a subsequent paper he and Stewart (**3**) proved the general result.

In this paper I shall prove the following related theorem.

Theorem. *If $\chi(r)$ is a real function with a continuous $n$-th derivative for all large $r$ and $f(z)$ is an integral function such that*

$$M(r) = M(r,f) \leqslant \chi(r)$$

*for all large $r$, then, given $\kappa > 1$, there is an infinite sequence of $r$ such that*

$$M_n(r) = M_n(r,f) < \kappa\chi^{(n)}(r).$$

The proof of the theorem depends so closely on $f(z)$ that it does not seem possible to extend it to show that the factor $n!(e/n)^n$ in (1) can be replaced by unity, as was conjectured by Hayman (**2**) in the cases $n = 1, 2$.

**2.** Since the theorem is easily shown to be true for polynomials, we assume in all that follows that $f(z)$ is not a polynomial. Let

$$f(z) = \sum_0^\infty a_\nu z^\nu$$

and write

$$F(z) = \sum_0^\infty \exp(\nu^{11/12}) z^\nu, \qquad \phi(z) = \sum_0^\infty a_\nu \exp(-\nu^{11/12}) z^\nu,$$

where $\phi$ is an integral function.

If $N = N(r,\phi)$ is the central index of $\phi(z)$ for $|z| = r$, then

$$|a_\nu|\exp(-\nu^{11/12}) r^\nu \leqslant |a_N|\exp(-N^{11/12}) r^N \quad (\nu = 0, 1, 2, \ldots), \tag{2}$$

and so $$\frac{|a_\nu|r^\nu}{|a_N|r^N} \leqslant \frac{\exp(\nu^{11/12})}{\exp(N^{11/12})} \quad (\nu = 0, 1, 2, \ldots).$$

Now choose $R < 1$ so that

$$\frac{d}{dx}(x^{11/12}+x\log R) = 0$$

for $x = N$, i.e. $$\tfrac{11}{12}N^{-1/12}+\log R = 0.$$

Then $N = N(R, F)$, and from (2) it follows that

$$\frac{|a_\nu|(rR)^\nu}{|a_N|(rR)^N} \leqslant \frac{\exp(\nu^{11/12})R^\nu}{\exp(N^{11/12})R^N} \leqslant 1, \tag{3}$$

and hence we get

$$N = N(r, \phi) = N(R, F) = N(rR, f).$$

Take $r_1 < r_2 < \ldots$ to be a sequence of values such that $r_\nu \to \infty$ as $\nu \to \infty$. Let $N_\nu = N(r_\nu, \phi)$ and denote by $R_\nu$ the value of $R$ such that

$$\tfrac{11}{12}N_\nu^{-1/12}+\log R = 0.$$

Since $N_\nu \to \infty$ as $\nu \to \infty$, it follows that $R_\nu \to 1$ as $\nu \to \infty$ and hence the sequence of values $r_\nu R_\nu$ ($= \rho_\nu$, say), which is denoted by $\beta$, increases to $\infty$. Unless otherwise stated $O(1)$ and $o(1)$ denote numbers which respectively remain bounded and tend to zero as $\rho_\nu \to \infty$.

We require several lemmas. The first two of these do in fact follow from a paper of mine (**1**), but as the results of this paper are not in a convenient form for quoting I have included proofs of them.

Lemma 1. *If $0 \geqslant \sigma \geqslant -N^{-12/13}$ then*

$$\left(\sum_{0}^{N-N^{7/12}} + \sum_{N+N^{7/12}}^{\infty}\right)\nu^q|a_\nu|(\rho e^\sigma)^\nu = o(1)N^q e^{N\sigma}M(\rho), \tag{4}$$

*where $q$ is a fixed positive integer and $N = N(\rho, f)$, as $\rho \to \infty$ through the sequence $\beta$.*

Suppose that $\rho = rR$ as in (3). Then

$$\sum_{0}^{N-N^{7/12}} \nu^q|a_\nu|(\rho e^\sigma)^\nu/\mu(\rho, f) \leqslant \sum_{0}^{N-N^{7/12}} \nu^q\exp(\nu^{11/12})(Re^\sigma)^\nu/\mu(R, F), \tag{5}$$

where $\mu(\rho, f)$ denotes the maximum term of $f(z)$ for $|z| = \rho$ and similarly for $\mu(R, F)$. It is not difficult to see that $\exp(x^{11/12})R^x$ increases steadily till $x = N$ and so, from (5),

$$\sum_{0}^{N-N^{7/12}} \nu^q|a_\nu|(\rho e^\sigma)^\nu/\mu(\rho, f) \leqslant N^{q+1}\exp(\xi^{11/12})R^\xi/\mu(R, F), \tag{6}$$

where $\xi = N-N^{7/12}$. Since

$$\xi^{11/12} = (N-N^{7/12})^{11/12} = N^{11/12}(1-\tfrac{11}{12}N^{-5/12}-\tfrac{11}{288}N^{-5/6})+O(1)$$

and $$R^{\xi} = \exp(\tfrac{11}{12}N^{\frac{1}{2}})R^N,$$
we see that
$$\exp(\xi^{11/12})R^{\xi}/\mu(R,F) = \exp\{-\tfrac{11}{288}N^{1/12}+O(1)\}. \tag{7}$$
From (6) and (7) it follows that
$$\begin{aligned}\sum_{0}^{N-N^{7/12}} \nu^q|a_\nu|(\rho e^{\sigma})^\nu/\mu(\rho,f) &= O(1)N^{q+1}\exp(-\tfrac{11}{288}N^{1/12}) \\ &= O(1)N^q e^{N\sigma}\exp(-\tfrac{11}{288}N^{1/12}+N^{1/13}+\log N) \\ &= o(1)N^q e^{N\sigma}. \end{aligned} \tag{8}$$

For the second sum in (4) we have, from (3),
$$\sum_{N+N^{7/12}}^{\infty} \nu^q|a_\nu|(\rho e^{\sigma})^\nu/\mu(\rho,f) \leqslant \Big(\sum_{N+N^{7/12}}^{3N}+\sum_{3N}^{\infty}\Big)\nu^q\exp(\nu^{11/12})(\rho e^{\sigma})^\nu/\mu(R,F). \tag{9}$$
Since $\exp(x^{11/12})R^x$ decreases steadily from $x = N$ and $\sigma \leqslant 0$, the first sum on the right-hand side of (9) is less than
$$(3N)^{q+1}e^{N\sigma}\exp(\xi^{11/12})R^{\xi}/\mu(R,F), \tag{10}$$
where $\xi = N+N^{7/12}$. Now
$$\xi^{11/12} = (N+N^{7/12})^{11/12} = N^{11/12}(1+\tfrac{11}{12}N^{-5/12}-\tfrac{11}{288}N^{-5/6})+O(1)$$
and $$R^{\xi} = \exp(-\tfrac{11}{12}N^{\frac{1}{2}})R^N,$$
and so $$\exp(\xi^{11/12})R^{\xi}/\mu(R,F) = \exp\{-\tfrac{11}{288}N^{1/12}+O(1)\}. \tag{11}$$
From (10) and (11) the first sum on the right-hand side of (9) is less than
$$O(1)N^q e^{N\sigma}\exp(-\tfrac{11}{288}N^{1/12}+\log N) = o(1)N^q e^{N\sigma}. \tag{12}$$

Let $$h(x) = x^{11/12}+x\log R,$$
and we get $$h(\nu) = h(N)+\int_N^{\nu}(\nu-\xi)h''(\xi)\,d\xi$$
since $h'(N) = 0$. Hence
$$h(\nu) = h(N)-\tfrac{11}{144}\int_N^{\nu}(\nu-\xi)\xi^{-13/12}\,d\xi$$
and putting $\xi = \nu\eta$ we obtain
$$h(\nu) = h(N)-\tfrac{11}{144}\nu^{11/12}\int_{N/\nu}^{1}(1-\eta)\eta^{-13/12}\,d\eta.$$
If $\nu \geqslant 3N$, then we have
$$h(\nu) < h(N)-\tfrac{11}{144}\nu^{11/12}\int_{1/3}^{1}(1-\eta)\,d\eta < h(N)-\tfrac{1}{72}\nu^{11/12}. \tag{13}$$
From (13) we see that the second sum on the right-hand side of (9) is less than
$$e^{N\sigma}\sum_{3N}^{\infty}\nu^q\exp(-\tfrac{1}{72}\nu^{11/12}) = O(1)e^{N\sigma} = o(1)N^q e^{N\sigma}. \tag{14}$$

From (9), (12), (14) and the fact that $\mu(\rho, f) < M(\rho)$, Lemma 1 follows.

**Lemma 2.** *If $0 \geqslant \sigma \geqslant -N^{-12/13}$ and $q$ is a fixed positive integer, then with $|z| = r$ and $N = N(r, f)$,*

$$\sum_{N-N^{7/12}}^{N+N^{7/12}} \{\nu(\nu-1)\ldots(\nu-q+1)e^{\nu\sigma} - N^q e^{N\sigma}\} a_\nu z^\nu = o(1) N^q e^{N\sigma} M(r) \quad (15)$$

*as $r \to \infty$.*

We write the sum in (15) as

$$\sum_{N-N^{7/12}}^{N+N^{7/12}} \{\nu(\nu-1)\ldots(\nu-q+1) - N^q\} e^{\nu\sigma} a_\nu z^\nu + N^q \sum_{N-N^{7/12}}^{N+N^{7/12}} (e^{\nu\sigma} - e^{N\sigma}) a_\nu z^\nu. \quad (16)$$

From Cauchy's inequality the first sum of (16) is less than

$$\left[\sum_{N-N^{7/12}}^{N+N^{7/12}} \{\nu(\nu-1)\ldots(\nu-q+1) - N^q\}^2 e^{2\nu\sigma}\right]^{\frac{1}{2}} \left[\sum_0^\infty |a_\nu|^2 r^{2\nu}\right]^{\frac{1}{2}}. \quad (17)$$

The first factor of (17) is less than

$$\begin{aligned} \exp\{(N - N^{7/12})\sigma\} \Big\{ \sum_{N-N^{7/12}}^{N+N^{7/12}} (\nu^q - N^q + O(1) N^{q-1})^2 \Big\}^{\frac{1}{2}} \\ &= O(1) e^{N\sigma} (N^{7/12} N^{2q-5/6})^{\frac{1}{2}} \\ &= O(1) N^{-1/8} N^q e^{N\sigma} \\ &= o(1) N^q e^{N\sigma}, \end{aligned} \quad (18)$$

where $O(1)$ and $o(1)$ refer to $r \to \infty$, and hence to $N \to \infty$. Also

$$\sum_0^\infty |a_\nu|^2 r^{2\nu} = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \leqslant M^2(r). \quad (19)$$

From (17), (18), (19) we get

$$\sum_{N-N^{7/12}}^{N+N^{7/12}} \{\nu(\nu-1)\ldots(\nu-q+1) - N^q\} e^{\nu\sigma} a_\nu z^\nu = o(1) N^q e^{N\sigma} M(r) \quad (20)$$

as $r \to \infty$.

The second sum of (16), by Cauchy's inequality, is less than

$$N^q \left[\sum_{N-N^{7/12}}^{N+N^{7/12}} (e^{\nu\sigma} - e^{N\sigma})^2\right]^{\frac{1}{2}} \left[\sum_0^\infty |a_\nu|^2 r^{2\nu}\right]^{\frac{1}{2}}. \quad (21)$$

The middle factor of (21) is

$$\begin{aligned} e^{N\sigma} \left[\sum_{N-N^{7/12}}^{N+N^{7/12}} \{e^{(\nu-N)\sigma} - 1\}^2\right]^{\frac{1}{2}} &= O(1) e^{N\sigma} \{N^{7/12} (N^{7/12} N^{-12/13})^2\}^{\frac{1}{2}} \\ &= O(1) e^{N\sigma} (N^{21/12 - 24/13})^{\frac{1}{2}} \\ &= O(1) e^{N\sigma} N^{-5/104} \\ &= o(1) e^{N\sigma}, \end{aligned} \quad (22)$$

where $O(1)$ and $o(1)$ refer to $r \to \infty$, and hence to $N \to \infty$. From (19), (21), (22) we see that

$$N^q \sum_{N-N^{7/12}}^{N+N^{7/12}} (e^{\nu\sigma}-e^{N\sigma})a_\nu z^\nu = o(1)N^q e^{N\sigma} M(r) \tag{23}$$

as $r \to \infty$.

Lemma 2 follows from (16), (19), (23).

Lemma 3. *Given any $\epsilon > 0$ and any positive integer $n$, there is an infinite sequence of $r$ such that*

$$1-\epsilon < \frac{r^n M_n(re^\sigma)}{N^n e^{N\sigma} M(r)} < 1+\epsilon \tag{24}$$

*for $0 \geqslant \sigma \geqslant N^{-12/13}$, where $N = N(r, f)$.*

Consider the expression

$$\begin{aligned}
&\sum_0^\infty \nu(\nu-1)...(\nu-n+1)a_\nu(ze^\sigma)^\nu - N^n e^{N\sigma} \sum_0^\infty a_\nu z^\nu \\
&= \Big(\sum_0^{N-N^{7/12}} + \sum_{N+N^{7/12}}^{\infty}\Big)\nu(\nu-1)...(\nu-n+1)a_\nu(ze^\sigma)^\nu - \\
&\qquad - \Big(\sum_0^{N-N^{7/12}} + \sum_{N+N^{7/12}}^{\infty}\Big)N^n e^{N\sigma} a_\nu z^\nu + \\
&\qquad + \sum_{N-N^{7/12}}^{N+N^{7/12}} \{\nu(\nu-1)...(\nu-n+1)e^{\nu\sigma} - N^n e^{N\sigma}\} a_\nu z^\nu.
\end{aligned} \tag{25}$$

From Lemma 1 we see that, if $|z| = r$ is a sufficiently large member of $\beta$, then each of the first two sums on the right-hand side of (25) is less than

$$\tfrac{1}{3}\epsilon N^n e^{N\sigma} M(r). \tag{26}$$

Also, if $|z| = r$ is sufficiently large, then, by Lemma 2, the last sum on the right-hand side of (25) is less than

$$\tfrac{1}{3}\epsilon N^n e^{N\sigma} M(r). \tag{27}$$

From (26) and (27) it follows that for all sufficiently large members of $\beta$ we have

$$\Big|\sum_0^\infty \nu(\nu-1)...(\nu-n+1)a_\nu(ze^\sigma)^\nu - N^n e^{N\sigma} \sum_0^\infty a^\nu z^\nu\Big| < \epsilon N^n e^{N\sigma} M(r). \tag{28}$$

If we now choose $z$ so that

$$\Big|\sum_0^\infty \nu(\nu-1)...(\nu-n+1)a_\nu(ze^\sigma)^\nu\Big| = r^n M_n(re^\sigma),$$

then (28) gives

$$r^n M_n(re^\sigma) - N^n e^{N\sigma} M(r) < \epsilon N^n e^{N\sigma} M(r),$$

which is equivalent to the right-hand inequality of (24). If $z$ is chosen so that

$$\left|\sum_{0}^{\infty} a_\nu z^\nu\right| = M(r),$$

then (28) gives

$$N^n e^{N\sigma} M(r) - r^n M_n(re^\sigma) < \epsilon N^n e^{N\sigma} M(r),$$

which is equivalent to the left-hand inequality of (24).

Thus Lemma 3 is proved.

LEMMA 4. *If $\alpha$ is a fixed (positive) number then*

$$\liminf_{r\to\infty} \frac{1}{(n-1)!} \int_{\alpha}^{r} (r-\rho)^{n-1} M_n(\rho)\, d\rho / M(r) = 1.$$

Let $r$ be a value for which Lemma 3 holds. Then

$$\begin{aligned}\frac{1}{(n-1)!} \int_{r\exp(-N^{-12/13})}^{r} & (r-\rho)^{n-1} M_n(\rho)\, d\rho \\ & < \frac{(1+\epsilon)N^n M(r)}{r^n (n-1)!} \int_{r\exp(-N^{-12/13})}^{r} (r-\rho)^{n-1} (\rho/r)^N\, d\rho \\ & < \frac{(1+\epsilon)N^n M(r)}{r^{n+N}} \frac{r^{N+n}}{(N+1)\dots(N+n)} \\ & < (1+\epsilon) M(r). \end{aligned} \tag{29}$$

Also, from Lemma 3,

$$\begin{aligned}\frac{1}{(n-1)!} \int_{\alpha}^{r\exp(-N^{-12/13})} & (r-\rho)^{n-1} M_n(\rho)\, d\rho \\ & < \frac{(1+\epsilon)N^n M(r)}{(n-1)!\, r^n} \exp(-N^{1/13}) \int_{0}^{r} (r-\rho)^{n-1}\, d\rho \\ & < \frac{(1+\epsilon)}{n!} N^n \exp(-N^{1/13}) M(r) \\ & < \epsilon M(r), \end{aligned} \tag{30}$$

if $r$ is taken large enough. Therefore, from (29) and (30), for all large $r$ for which Lemma 3 is true, we get

$$\frac{1}{(n-1)!} \int_{\alpha}^{r} (r-\rho)^{n-1} M_n(\rho)\, d\rho < (1+2\epsilon) M(r). \tag{31}$$

Further

$$f(z) = \frac{1}{(n-1)!}\int\limits_0^z (z-\zeta)^{n-1} f^{(n)}(\zeta)\,d\zeta - \frac{f^{(n-1)}(0)}{(n-1)!}z^{n-1} + \ldots + f(0),$$

in which the integral is taken along a radius joining the origin to a point of $|z| = r$, where $|f(z)| = M(r)$. Using this expression we obtain

$$M(r) < \frac{1}{(n-1)!}\int\limits_\alpha^r (r-\rho)^{n-1} M_n(\rho)\,d\rho + O(1)r^{n-1} \tag{32}$$

as $r \to \infty$. From (31) and (32), Lemma 4 follows.

**3.** Suppose the theorem false. Then there is an integral function $f(z)$ and a real function $\chi(r)$ with a continuous $n$th derivative such that, for all large $r$,

$$M(r) \leqslant \chi(r),$$

and such that for some $\eta > 0$ we have

$$M_n(r) > (1+\eta)\chi^{(n)}(r)$$

for $r \geqslant r_0$. Hence, for $r \geqslant r_0$,

$$\begin{aligned}\frac{1}{(n-1)!}\int\limits_{r_0}^r (r-\rho)^{n-1} M_n(\rho)\,d\rho &> \frac{1+\eta}{(n-1)!}\int\limits_{r_0}^r (r-\rho)^{n+1}\chi^{(n)}(\rho)\,d\rho \\ &> (1+\eta)\chi(r) - O(1)r^{n-1}\end{aligned}$$

as $r \to \infty$. Therefore for this function $f(z)$ we get

$$\begin{aligned}&\liminf_{r\to\infty} \frac{1}{(n-1)!}\int\limits_{r_0}^r (r-\rho)^{n-1} M_n(\rho)\,d\rho / M(r) \\ &\qquad\qquad \geqslant \liminf_{r\to\infty} (1+\eta)\chi(r)/M(r) \geqslant 1+\eta.\end{aligned}$$

This contradicts Lemma 4 and so the theorem is true.

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# THE GROUPS $\pi_r(V_{n,m})$ (II)

*By* G. F. PAECHTER (*Oxford*)

[Received 20 November 1956]

## Introduction

THIS is the second of a sequence of five papers, the first being (4), in which I calculate certain homotopy groups of the Stiefel manifolds $V_{n,m}$. The present paper contains the calculations of those $\pi_r(V_{k+3,3})$ which are given in the following table. There $\pi^p_{k,m}$ denotes $\pi_{k+p}(V_{k+m,m})$, $Z_q$ a cyclic group of order $q$, and $+$ direct summation. Also $s > 1$.† In the Appendix I examine the homotopy type of certain reduced projective spaces $P^n_k$ which are required in this and the subsequent papers. Please note that paragraphs are numbered consecutively throughout the whole sequence of papers, §§ 1–5 being contained in (I), §§ 6–7 in (II), § 8 in (III), § 9 in (IV), and §§ 10–13 in (V).

TABLE FOR $\pi^p_{k,3}$‡

| $k$ | $p=1$ | $p=2$ | $p=3$ | $p=4$ | $p=5$ | $p=6$ | $p=7$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | $Z_\infty+Z_\infty$ | $Z_2+Z_2$ | $Z_2+Z_2$ | $Z_{12}+Z_{12}$ | $Z_2+Z_2$ | $Z_2+Z_2$ |
| 3 | 0 | $Z_2+Z_\infty$ | $Z_2$ | $Z_\infty+Z_4$ | $Z_2+Z_{24}$ | | |
| 4 | $Z_2+Z_2$ | $Z_2+Z_2$ | $Z_\infty+Z_{12}+Z_4$ | $Z_2+Z_2+Z_2+Z_2$ | $Z_2+Z_2+Z_2+Z_2$ | | |
| 5 | $Z_2$ | $Z_4+Z_\infty$ | $Z_2+Z_2+Z_2$ | $Z_2+Z_2+Z_2$ | $Z_{24}$ | | |
| 6 | $Z_4$ | $Z_2$ | $Z_{12}+Z_2$ | $Z_2$ | $Z_\infty+Z_2$ | | |
| $4s-1$ | 0 | $Z_2+Z_\infty$ | $Z_2+Z_2$ | $Z_4$ | $Z_{24}$ | | |
| $4s+1$ | $Z_2$ | $Z_4+Z_\infty$ | $Z_2+Z_2+Z_2$ | $Z_2+Z_2$ | $Z_{24}$ | | |
| $4s$ | $Z_2+Z_2$ | $Z_2+Z_2$ | $Z_{24}+Z_4$ | $Z_2+Z_2$ | $Z_2$ | | |
| $4s+2$ | $Z_4$ | $Z_2$ | $Z_{12}+Z_2$ | $Z_2$ | $Z_2$ | | |

## 6. The construction '$Q^r$'

In § 7.1, I shall for the first time make use of the following construction, which is isolated here for easier subsequent reference.§

Let $r > 1$ and $r \equiv 1 \pmod 2$. Let $S^r_1$ and $S^r_2$ be given in $R^{r+1}$ by the respective equations

$$x_0^2+x_1^2+\ldots+x_r^2 = 1, \qquad x_0^2+x_1^2+\ldots+2x_r^2 = 1.$$

Let $$S^{r-1} = S^r_1 \cap S^r_2, \qquad Q^r = S^r_1 \cup S^r_2.$$

For $i = 1$ or $2$ let $E^r_{i+}$ and $E^r_{i-}$ be the hemispheres of $S^r_i$ for which respectively $x_r \geqslant 0$ and $x_r \leqslant 0$, oriented in accordance with $S^r_i$ and in

† For a full table of results and for references see (4) 249.

‡ Since $\pi^p_{2,m} \approx \pi^{p+1}_{1,m+1}$ by Theorem 4.2 (*a*), the values of $\pi^p_{2,3}$ are obtained with those of $\pi^{p+1}_{1,4}$ in the third paper.

§ This is essentially the construction on p. 270 of (7).

such a way that $E_{1+}^r$ and $E_{2+}^r$ induce opposite orientations of $S^{r-1}$. Thus the radial projection $S_2^r \to S_1^r$ is of degree $-1$, and $(E_{1+}^r \cup E_{2+}^r)$ and $(E_{1-}^r \cup E_{2-}^r)$ are both oriented $r$-spheres.

Clearly $\pi_s(Q^r) = 0$ if $s < r$. Hence by a theorem due to Hurewicz [(3) Theorem 1] there is an isomorphism $\pi_r(Q^r) \approx H_r(Q^r)$. Thus, if

$$\alpha_1, \alpha_2, \beta_1, \beta_2, \in \pi_r(Q^r),$$

are classes of the identical maps of

$$S_1^r, S_2^r, (E_{1+}^r \cup E_{2+}^r), (E_{1-}^r \cup E_{2-}^r),$$

then we have that $$\alpha_1+\alpha_2 = \beta_1+\beta_2.$$

Now let $f_i$: $S_i^r \to X$ ($i = 1, 2$; $X$ $r$-simple) be symmetric maps which agree on $S^{r-1}$. Let

$$h_1: (E_{1+}^r \cup E_{2+}^r) \to X \text{ be defined by } h_1 = f_i \text{ on } E_{i+}^r,$$

$$h_2: (E_{1-}^r \cup E_{2-}^r) \to X \text{ be defined by } h_2 = f_i \text{ on } E_{i-}^r.$$

Let $H$: $Q^r \to X$ be defined by $h_1$ on $(E_{1+}^r \cup E_{2+}^r)$ and $h_2$ on $(E_{1-}^r \cup E_{2-}^r)$. Then $H$ induces a homomorphism $H_*$: $\pi_r(Q^r) \to \pi_r(X)$, and $H_* \alpha_i = \{f_i\}$, $H_* \beta_i = \{h_i\}$. But $\{h_1\} = \{h_2\} = \{h\}$, say, since the $f_i$ are symmetric maps and $r \equiv 1 \pmod 2$. Hence we have

$$\{f_1\}+\{f_2\} = 2\{h\}.$$

Together with this construction we need the following theorem on symmetric maps of spheres. Let $S^r$, $P^{r-1}$, and $u_{r-1}$ be defined as in § 2.3 (c), $u_{r-1}^{-1}(P^{r-1})$ being the equator of $S^r$. Then we have

THEOREM 6.1. *No symmetric map* $f$: $S^r \to S^{r-1}$ *is essential unless* $r \equiv 3 \pmod 4$. *If* $r \equiv 3 \pmod 4$ *and* $fu_{r-1}^{-1}$: $P^{r-1} \to S^{r-1}$ *is essential, then* $f$ *is essential.*

For the proof see J. H. C. Whitehead (7) Theorem 7.

## 7. Calculation of $\pi_{k,3}^p$†

We consider the fibring $V_{k+3,3}/V_{k+2,2} \to S^{k+2}$, and examine the sequence

$$\text{(B)} \qquad \to \pi_{k+p+1}(S^{k+2}) \xrightarrow{\Delta_*} \pi_{k,2}^p \xrightarrow{i_{k+p*}} \pi_{k,3}^p \xrightarrow{p_{k+p*}} \pi_{k+p}(S^{k+2}) \to.$$

7.1. $k \equiv 1 \pmod 4$. In this case there is a two-field on $S^{k+2}$ (2, 5) and so the fibring admits a cross-section $\mathfrak{p}$. Hence Theorem 1.1 gives

$$\pi_{k,3}^p = i_* \pi_{k,2}^p + \mathfrak{p}_* \pi_{k+p}(S^{k+2}).$$

Using the values of $\pi_{k,2}^p$ as calculated in § 5.2, we obtain the values shown in the table for $\pi_{k,3}^p$ when $k \equiv 1 \pmod 4$.

† For the notation used see (4), especially §§ 2 and 3.1.

Note that, by Theorem 1.2 and Corollary 1.5, we have

$$\{t_{k+3,3}\} = 0 \quad \textit{for } k \equiv 1 \pmod 4.$$

7.2. $k \equiv 3 \pmod 4$

(*a*) When $p = 1$, (B) gives

$$\xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+2}) \xrightarrow{\Delta_*} \pi^1_{k,2} \xrightarrow{i_{k+1*}} \pi^1_{k,3} \to \pi_{k+1}(S^{k+2}),$$

i.e.

$$\to Z_\infty \to Z_2 \to \pi^1_{k,3} \to 0,$$

by § 5.2 (*b*). But $i^{-1}_{k+1*}(0) \neq 0$, for otherwise we should have a cross-section in the fibring by Theorem 1.2, and so a two-field on $S^{k+2}$ which is impossible (**2**, **5**). Hence $i^{-1}_{k+1*}(0) = \pi^1_{k,2}$, and

$$\pi^1_{k,3} = 0.$$

Note that $\Delta_*$ is onto, whence the image of $p_{k+2*}$ is the $Z_\infty$ subgroup generated by $2\{h_{k+2,k+2}\}$. We also have from Corollary 1.5 that $\{t_{k+3,3}\}$ generates $i^{-1}_{k+1*}(0)$.

Hence $$\{t_{k+3,3}\} = \{i_{k+1,1}\, h_{k,k+1}\}.$$

(*b*) When $p = 2$, (B) gives

$$\xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+2}) \xrightarrow{\Delta_*} \pi^2_{k,2} \xrightarrow{i_{k+2*}} \pi^2_{k,3} \xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+2}) \to,$$

i.e.

$$\to Z_2 \to Z_4 \to \pi^2_{k,3} \to Z_\infty \to 0,$$

by § 5.2 (*c*) and since the image of $p_{k+2*}$ is $Z_\infty$ by (*a*). Also

$$i^{-1}_{k+2*}(0) = t_{k+3,3*}\, \pi_{k+2}(S^{k+1}),$$

which is generated by

$$h^*_{k+1,k+2}\{t_{k+3,3}\} = h^*_{k+1,k+2}\{i_{k+1,1}\, h_{k,k+1}\} = \{i_{k+1,1}\, h_{k,k+2}\},$$

which is non-zero and of order two by § 5.2 (*c*). Thus

$$\pi^2_{k,3} = Z_2 + Z_\infty,$$

and is generated by $i_{k+2,1*}\, a$ of order two, where

$$p_{k+2,1*}\, a = \{h_{k+1,k+2}\},$$

and $b$, of infinite order, which is such that $p_{k+3,1*}\, b = 2\{h_{k+2,k+2}\}$. Note also that the image of $\Delta_* = i^{-1}_{k+2*}(0)$ is $Z_2$, and thus that the image of $p_{k+3*}$ is zero.

(*c*) When $p = 3$ and $k \geqslant 7$, (B) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+2}) \xrightarrow{\Delta_*} \pi^3_{k,2} \xrightarrow{i_{k+3*}} \pi^3_{k,3} \xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+2}) \to,$$

i.e.

$$\to Z_2 \to Z_2 + Z_2 \to \pi^3_{k,3} \to 0$$

by § 5.2 (*d*) and since $p_{k+3*}$ is trivial by (*b*). Also

$$i_{k+3*}^{-1}(0) = t_{k+3,3*}\pi_{k+3}(S^{k+1}),$$

which is generated by

$$h_{k+1,k+3}^{*}\{t_{k+3,3}\} = h_{k+1,k+3}^{*}\{i_{k+1,1}h_{k,k+1}\} = \{i_{k+1,1}(12h_{k,k+3})\} = 0, \text{by } 5.2\,(d).$$

Hence $$i_{k+3*}^{-1}(0) = 0,$$

and $$\pi_{k,3}^{3} = Z_2 + Z_2,$$

being generated by $\{i_{k+1,2}h_{k,k+3}\}$ and $i_{k+2,1*}a$, where

$$p_{k+2,1*}a = \{h_{k+1,k+3}\}.$$

Note also that since $\Delta_*$ is trivial, $p_{k+4*}$ is onto.

(*d*) When $p = 3$ and $k = 3$, (B) gives

$$\xrightarrow{p_{7*}} \pi_7(S^5) \xrightarrow{\Delta_*} \pi_{3,2}^{3} \xrightarrow{i_{6*}} \pi_{3,3}^{3} \xrightarrow{p_{6*}} \pi_6(S^5) \to,$$

i.e. $$\to Z_2 \to Z_2 \to \pi_{3,3}^{3} \to 0,$$

by § 5.2 (*e*) and since $p_{6*}$ is trivial by (*b*). Also $i_{6*}^{-1}(0) = t_{6,3*}\pi_6(S^4)$, which is generated by

$$h_{4,6}^{*}\{t_{6,3}\} = h_{4,6}^{*}\{i_{4,1}h_{3,4}\} = \{i_{4,1}(6h_{3,6})\} = 0,$$

by § 5.2 (*e*). Hence $$i_{6*}^{-1}(0) = 0,$$

and $$\pi_{3,3}^{3} = Z_2,$$

being generated by $i_{5,1*}a$, where $p_{5,1*}a = \{h_{4,6}\}$. Note also that since $\Delta_*$ is trivial, $p_{7*}$ is onto $\pi_7(S^5)$.

(*e*) When $p = 4$ and $k \geqslant 7$, (B) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+2}) \xrightarrow{\Delta_*} \pi_{k,2}^{4} \xrightarrow{i_{k+4*}} \pi_{k,3}^{4} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+2}) \to,$$

i.e. $$\to Z_{24} \to Z_2 \to \pi_{k,3}^{4} \to Z_2 \to 0,$$

by § 5.2 (*f*) and since $p_{k+4*}$ is onto $\pi_{k+4}(S^{k+2})$ by (*c*). Also

$$i_{k+4*}^{-1}(0) = t_{k+3,3*}\pi_{k+4}(S^{k+1}),$$

which is generated by

$$h_{k+1,k+4}^{*}\{t_{k+3,3}\} = h_{k+1,k+4}^{*}\{i_{k+1,1}h_{k,k+1}\} \in i_{k+1,1*}\pi_{k+4}(S^k) = 0.$$

Hence $$i_{k+4*}^{-1}(0) = 0,$$

and $\pi_{k,3}^{4}$ *has four elements.*

Note also that $\Delta_*$ is trivial and that thus $p_{k+5*}$ is onto.

To determine the structure of $\pi_{k,3}^{4}$, consider the sequence associated with the fibring $$V_{4s+2,3}/S^{4s-1} \to V_{4s+2,2},$$

i.e. $$\to \pi_{4s+p}(S^{4s-1}) \to \pi_{4s-1,3}^{p+1} \to \pi_{4s,2}^{p} \xrightarrow{\Delta_*} \pi_{4s+p-1}(S^{4s-1}) \to,$$

which, starting at the term $\pi_{4s+3}(S^{4s-1})$, is, for $s \geqslant 2$, of the form

$$\to 0 \to \pi^4_{4s-1,3} \to Z_{24}+Z_2 \xrightarrow{\Delta_*} Z_{24} \to Z_2+Z_2 \to Z_2+Z_2 \xrightarrow{\Delta_*} Z_2 \to,$$

by virtue of the results of § 5.1 and (c). But, since $\pi^4_{4s-1,3}$ has four elements, we have, by working along the sequence from the left, that

$$\text{(i)}\quad \Delta_* \pi^3_{4s,2} = 2\pi_{4s+2}(S^{4s-1}),$$

$$\text{(ii)}\quad \Delta_* \pi^2_{4s,2} = \pi_{4s+1}(S^{4s-1}).$$

We now operate with $h^* = \{h_{4s+2,4s+3}, h_{4s+1,4s+2}\}^*$ on the portion

$$\to \pi^2_{4s,2} \xrightarrow{\Delta_*} \pi_{4s+1}(S^{4s-1}) \to$$

of the above sequence to obtain the commutative diagram (Lemma 3.1)

$$\begin{tikzcd} 0 \arrow[r] & \pi^4_{4s-1,3} \arrow[r] & \pi^3_{4s,2} \arrow[r, "\Delta_*"] & \pi_{4s+2}(S^{4s-1}) \arrow[r] & \pi^3_{4s-1,3} \arrow[r] & {} \\ & \arrow[r] & \pi^2_{4s,2} \arrow[u, "h^*"] \arrow[r, "\Delta_*"] & \pi_{4s+1}(S^{4s-1}) \arrow[u, "h^*"] \arrow[r] & . \end{tikzcd}$$

By virtue of the above relations this takes the form

$$\begin{tikzcd} 0 \arrow[r] & \pi^4_{4s-1,3} \arrow[r] & Z_{24}+Z_2 \arrow[r, "\Delta_*"] & Z_{24} \arrow[r] & Z_2 \subset Z_2+Z_2 \arrow[r] & 0 \\ & \arrow[r] & Z_2+Z_2 \arrow[u, "h^*"] \arrow[r, "\Delta_*"] & Z_2 \arrow[u, "h^*"] \arrow[r] & 0. \end{tikzcd}$$

Using the result of § 5.1, now choose generators of $\pi^2_{4s,2}$, $= Z_2+Z_2$, as

$$a = i_*\{h_{4s,4s+2}\} \quad\text{and}\quad b = \mathfrak{p}_*\{h_{4s+1,4s+2}\};$$

and of $\pi^3_{4s,2}$, $= Z_{24}+Z_2$, as

$$c = i_*\{h_{4s,4s+3}\} \quad\text{and}\quad d = p_*\{h_{4s+1,4s+3}\}.$$

Then we have, by Lemma 3.1, that

$$h^*a = h^*i_*\{h_{4s,4s+2}\} = i_* h^*_{4s+2,4s+3}\{h_{4s,4s+2}\} = i_*\{12h_{4s,4s+3}\} = 12c,$$

$$h^*b = h^*\mathfrak{p}_*\{h_{4s+1,4s+2}\} = \mathfrak{p}_* h^*_{4s+2,4s+3}\{h_{4s+1,4s+2}\} = \mathfrak{p}_*\{h_{4s+1,4s+3}\} = d.$$

But *either* $\Delta_* a$ *or* $\Delta_* b = \{h_{4s-1,4s+1}\}$, *or both.*

Hence

*either* $h^*\Delta_* a$ *or* $h^*\Delta_* b = 12\{h_{4s-1,4s+2}\}$, *or both,*

i.e. *either* $\Delta_* 12c$ *or* $\Delta_* d = 12\{h_{4s-1,4s+2}\} \neq 0$, *or both.*

But, if $\pi^4_{4s-1,3}$ were $Z_2+Z_2$, its (isomorphic) image in $Z_{24}+Z_2$ would be generated by $12c$ and $d$. Then both $\Delta_* 12c$ and $\Delta_* d$ would be zero by exactness, which we have just seen to be impossible.

Hence

$$\pi^4_{k,3} = Z_4 \quad \textit{when } k \equiv 3 \pmod 4) \textit{ and when } k \geqslant 7,$$

and is generated by an element $a$ such that $p_{k+3,1*}a = \{h_{k+2,k+4}\}$. Note that it is $\Delta_* d$ which equals $12\{h_{4s-1,4s+2}\} \neq 0$.

(*f*) When $p = 4$ and $k = 3$, (B) gives

$$\xrightarrow{p_{8*}} \pi_8(S^5) \xrightarrow{\Delta_*} \pi^4_{3,2} \xrightarrow{i_{7*}} \pi^4_{3,3} \xrightarrow{p_{7*}} \pi_7(S^5) \rightarrow,$$

i.e.

$$\rightarrow Z_{24} \rightarrow Z_\infty + Z_2 \rightarrow \pi^4_{3,3} \rightarrow Z_2 \rightarrow 0,$$

by § 5.2 (*g*) and since $p_{7*}$ is onto by (*d*). Also

$$\begin{aligned} i^{-1}_{7*}(0) = t_{6,3*}\pi_7(S^4) = \{i_{4,1}h_{3,4}\}_*\pi_7(S^4) &= i_{7*}\{h_{3,4}\}_*\pi_7(S^4) \\ &\subset i_{7*}\pi_7(S^3) = 0, \quad \text{by 5.2 } (g). \end{aligned}$$

Hence $\qquad \pi^4_{3,3}$ *is an extension of* $Z_\infty + Z_2$ *by* $Z_2$.

Note that $\Delta_*$ is trivial, whence $p_{8*}$ is onto $\pi_8(S^5)$.

There are now three possibilities for the structure of $\pi^4_{3,3}$: $Z_\infty + Z_2$, $Z_\infty + Z_2 + Z_2$, $Z_\infty + Z_4$. I shall prove that it is the last by eliminating the other two. As in (*e*) we consider the sequence associated with the fibring $V_{6,3}/S^3 \rightarrow V_{6,2}$, i.e.

$$\rightarrow \pi_{p+4}(S^3) \xrightarrow{i_{4,2*}} \pi^{p+1}_{3,3} \xrightarrow{p_{6,2*}} \pi^p_{4,2} \xrightarrow{\Delta_*} \pi_{p+3}(S^3) \rightarrow,$$

which, starting at the term $\pi_7(S^3)$, is of the form

$$\rightarrow Z_2 \rightarrow \pi^4_{3,3} \rightarrow Z_\infty + Z_{12} + Z_2 \xrightarrow{\Delta_*} Z_{12} \rightarrow Z_2 \rightarrow Z_2 + Z_2 \xrightarrow{\Delta_*} Z_2 \rightarrow,$$

by virtue of the results of § 5.1 and (*d*). Thus, working from the right, we have by exactness, that

$$\begin{aligned} &\text{(i)} \quad \Delta_*\pi^2_{4,2} = \pi_5(S^3), \\ &\text{(ii)} \quad i_{4,2*}\pi_6(S^3) = 0, \\ &\text{(iii)} \quad \Delta_*\pi^3_{4,2} = \pi_6(S^3). \end{aligned}$$

We now operate with $h^*_{6,7}$ on the portion

$$\rightarrow \pi_6(S^3) \xrightarrow{i_{4,2*}} \pi^3_{3,3} \rightarrow$$

of the above sequence to obtain the commutative diagram (Lemma 3.1)

$$\begin{tikzcd} \rightarrow \pi_7(S^3) \arrow[r, "i_{4,2*}"] & \pi^4_{3,3} \arrow[r, "p_{6,2*}"] & \pi^3_{4,2} \rightarrow \\ \rightarrow \pi_6(S^3) \arrow[u, "h^*"] \arrow[r, "i_{4,2*}"] & \pi^3_{3,3} \arrow[u, "h^*"] \arrow[r, "p_{6,2*}"] & . \end{tikzcd}$$

Thus $\qquad i_{4,2*}\pi_7(S^3) = i_{4,2*}h^*\pi_6(S^3) = h^*i_{4,2*}\pi_6(S^3) = 0,$

by (ii) above. Hence $p_{6,2*}: \pi^4_{3,3} \rightarrow \pi^3_{4,2}$ is a monomorphism.

Next we operate with $h^* = \{h_{6,7}, h_{5,6}\}^*$ on the portion

$$\to \pi^2_{4,2} \xrightarrow{\Delta_*} \pi_5(S^3) \to$$

of the sequence to obtain the commutative diagram

$$\begin{tikzcd}
\to \pi^4_{3,3} \arrow[r, "p_{6,2*}"] & \pi^3_{4,2} \arrow[r, "\Delta_*"] & \pi_6(S^3) \to \\
\to & \pi^2_{4,2} \arrow[u, "h^*"'] \arrow[r, "\Delta_*"] & \pi_5(S^3) \to . \arrow[u, "h^*"']
\end{tikzcd}$$

By virtue of the above relations this takes the form

$$\begin{tikzcd}
0 \to \pi^4_{3,3} \arrow[r] & Z_\infty + Z_{12} + Z_2 \arrow[r] & Z_{12} \to 0 \\
\to & Z_2 + Z_2 \arrow[u, "h^*"'] \arrow[r] & Z_2 \to 0. \arrow[u, "h^*"']
\end{tikzcd}$$

Using the result of § 5.1, now choose generators of $\pi^2_{4,2}$, $= Z_2 + Z_2$, as

$$a = i_*\{h_{4,6}\} \quad \text{and} \quad b = \mathfrak{p}_*\{h_{5,6}\};$$

and of the finite summand of $\pi^3_{4,2}$, $= Z_\infty + Z_{12} + Z_2$, as

$$c = i_*\,\mathfrak{E}\{h_{3,6}\} \quad \text{and} \quad d = \mathfrak{p}_*\{h_{5,7}\}.$$

Then, as in (*e*), we have by Lemma 3.1 that

$$h^*a = h^*i_*\{h_{4,6}\} = i_*h^*_{6,7}\{h_{4,6}\} = i_*\,6\mathfrak{E}\{h_{3,6}\} = 6c,$$

and $$h^*b = h^*\mathfrak{p}_*\{h_{5,6}\} = \mathfrak{p}_*h^*_{6,7}\{h_{5,6}\} = \mathfrak{p}_*\{h_{5,7}\} = d.$$

But *either* $\Delta_* a$ *or* $\Delta_* b = \{h_{3,5}\}$, *or both.*

Hence *either* $h^*\Delta_* a$ *or* $h^*\Delta_* b = 6\{h_{3,6}\}$, *or both,*

i.e. *either* $\Delta_* 6c$ *or* $\Delta_* d = 6\{h_{3,6}\} \neq 0$, *or both.*

But, if $\pi^4_{3,3}$ were $Z_\infty + Z_2 + Z_2$, the (isomorphic) image of its finite subgroup in $Z_\infty + Z_{12} + Z_2$ would be generated by $6c$ and $d$. Then both $\Delta_* 6c$ and $\Delta_* d$ would be zero by exactness, which we have just seen to be impossible. So $\pi^4_{3,3}$ cannot be $Z_\infty + Z_2 + Z_2$.

Now consider the diagram, commutative by § 2.1,

$$\begin{tikzcd}
0 \to \pi^4_{3,3} \arrow[r, "p_{6,2*}"] & \pi^3_{4,2} \arrow[r, "\Delta_*"] & \pi_6(S^3) \to 0 \\
\to \pi^4_{3,2} \arrow[u, "i_{5,1*}"'] \arrow[r, "p_{5,1*}"] & \pi_7(S^4) \arrow[u, "i_{5,1*}"'] \arrow[r, "\Delta_*"] & \pi_6(S^3) \to, \arrow[u, "i_{4,0*}"']
\end{tikzcd}$$

where the upper sequence is that of the previous paragraph and the lower that associated with the fibring $V_{5,2}/S^3 \to S^4$. Then, in the notation of

the previous paragraph, we have that

$$\Delta_* 6c = \Delta_* 6i_{5,1*}\,\mathfrak{E}\{h_{3,6}\} = i_{4,0*}\,\Delta_* 6\mathfrak{E}\{h_{3,6}\} = i_{4,0*}\,12\{h_{3,6}\},$$

by § 5.2 (*e*), $= 0$. Assume now that $\pi^4_{3,3} = Z_\infty + Z_2$, and let

$$G = \pi^3_{4,2}/p_{6,2*}\,\pi^4_{3,3}.$$

We know that $G$ is of order 12. For this to be so the image under $p_{6,2*}$ of the infinite summand of $\pi^4_{3,3}$ must be an infinite summand of $\pi^3_{4,2}$. However, $d$ is of order 2, and we have just seen that $\Delta_* 6c = 0$. Thus $G$ can have no element of order 12. Since we know that in fact $G = Z_{12}$, we have arrived at a contradiction. Thus $\pi^4_{3,3}$ cannot be $Z_\infty + Z_2$.

Hence $$\pi^4_{3,3} = Z_\infty + Z_4,$$

and is generated by

$$i_{5,1*}\,p^{-1}_{5,1*}(2\bar{p}_*\{h_{7,7}\} - \lambda\mathfrak{E}\{h_{3,6}\}),$$

where $\lambda$ is odd, and $a$, of order four, such that $p_{6,1*}a = \{h_{5,7}\}$. Note that $2a = i_{5,1*}\,p^{-1}_{5,1*}\,6\mathfrak{E}\{h_{3,6}\}$.

(*g*) When $p = 5$, and $k \geqslant 7$, (B) gives

$$\to \pi_{k+6}(S^{k+2}) \to \pi^5_{k,2} \to \pi^5_{k,3} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+2}) \to,$$

i.e. $$\to 0 \to 0 \to \pi^5_{k,3} \to Z_{24} \to 0,$$

by § 5.2 (*i*) and since $p_{k+5*}$ is onto by (*e*).

Thus $\pi^5_{k,3} = Z_{24}$ *for* $k \equiv 3 \pmod 4$ *and for* $k \geqslant 7$,

and is generated by $p^{-1}_{k+3,1*}\{h_{k+2,k+5}\}$.

(*h*) When $p = 5$ and $k = 3$, (B) gives

$$\xrightarrow{p_{9*}} \pi_9(S^5) \xrightarrow{\Delta_*} \pi^5_{3,2} \xrightarrow{i_{8*}} \pi^5_{3,3} \xrightarrow{p_{8*}} \pi_8(S^5) \to,$$

i.e. $$\to Z_2 \to Z_2 \to \pi^5_{3,3} \to Z_{24} \to 0,$$

by § 5.2 (*j*) and since $p_{8*}$ is onto by (*f*). Also

$$i^{-1}_{8*}(0) = t_{6,3*}\,\pi_8(S^4) = \{i_{4,1}\,h_{3,4}\}_*\,\pi_8(S^4) = i_{8*}\{h_{3,4}\}_*\,\pi_8(S^4)$$
$$\subset i_{8*}\,\pi_8(S^3), \quad \text{which is zero by 5.2 } (j).$$

Hence $\pi^5_{3,3}$ *is an extension of* $Z_2$ *by* $Z_{24}$.

Note that $\Delta_*$ is trivial, whence $p_{9*}$ is onto $\pi_9(S^5)$.

To calculate the extension we consider the sequence associated with the fibring $V_{6,3}/S^3 \to V_{6,2}$, which contains the portion

$$\to \pi^5_{3,3} \xrightarrow{p_*} \pi^4_{4,2} \xrightarrow{\Delta_*} \pi_7(S^3) \to .$$

By virtue of the results in § 5.1 this is of the form

$$\to \pi^5_{3,3} \xrightarrow{p_*} Z_2 + Z_2 + Z_{24} \xrightarrow{\Delta_*} Z_2 \to .$$

Since $\pi^5_{3,3}$ is of order 48, we see that $p_*$ is a monomorphism, whilst $\Delta_*$ is onto. But it is impossible to map $Z_{48}$ isomorphically into $Z_2+Z_2+Z_{24}$.

Hence $$\pi^5_{3,3} = Z_2+Z_{24},$$
and is generated by $i_{5,1*}p^{-1}_{5,1*}\mathfrak{E}\{h_{3,7}\}$, and $a$ such that $p_{6,1*}a = \{h_{5,8}\}$.

7.3. $k \equiv 0 \pmod 2$ *and* $\geqslant 4$. Our first task is to calculate
$$\{t_{k+3,3}\} \in \pi^1_{k,2} = Z_2+Z_\infty$$
by § 5.1. We have by (iii) of § 2.4 (*b*) that $t_{k+2,2} \mid S^k = i_{k+1,1}t_{k+2,2}$; and by § 5.1 that $\{t_{k+2,2}\} = 0$. Thus we can extend $i_{k+1,1}t_{k+2,2}$ over the hemisphere $E^{k+1}_+$ of $S^{k+1}$, and, since $t_{k+2,2}$ is a symmetric map [§ 2.4 (*a*)], we can extend it symmetrically over $E^{k+1}_-$. Denote this extension by
$$g\colon S^{k+1} \to i_{k+1,1}(S^k) \subset V_{k+2,2}.$$

We now use the construction '$Q^r$' of § 6 with
$$r = k+1, \qquad X = V_{k+2,2}, \qquad f_1 = t_{k+3,3}, \qquad f_2 = g$$
as defined above. Then we have that
$$2\{h\} = \{f_1\}+\{f_2\} = \{t_{k+3,3}\}+\{g\}.$$
Hence
$$p_{k+2,1*}2\{h\} = p_{k+2,1*}\{t_{k+3,3}\}+p_{k+2,1*}\{g\}$$
$$= 2\{h_{k+1,k+1}\}, \text{ by 2.3 } (b) \text{ and since } \{g\} \in i_*\pi_{k+1}(S^k).$$
Thus
$$p_{k+2,1*}\{h\} = \{h_{k+1,k+1}\}$$
and $$\{h\} = \{\mathrm{p}h_{k+1,k+1}\}+i_*w, \quad \text{where } w \in \pi_{k+1}(S^k).$$
Thus $$2\{h\} = 2\{\mathrm{p}h_{k+1,k+1}\},$$
whence $$\{t_{k+3,3}\} = 2\{\mathrm{p}h_{k+1,k+1}\}-\{g\}.$$

7.31. $k \equiv 0 \pmod 4$. We have from § 7.3 that
$$\{t_{k+3,3}\} = 2\{\mathrm{p}h_{k+1,k+1}\}-\{g\}.$$
But $g\colon S^{k+1} \to S^k$ is a symmetric map by construction, and so is inessential by Theorem 6.1. Thus
$$\{t_{k+3,3}\} = 2\{\mathrm{p}h_{k+1,k+1}\}.$$
We are now ready to examine the sequence (B).

(*a*) When $p = 1$, (B) gives
$$\xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+2}) \xrightarrow{\Delta_*} \pi^1_{k,2} \xrightarrow{i_{k+1*}} \pi^1_{k,3} \to \pi_{k+1}(S^{k+2}),$$
i.e. $$\to Z_\infty \to Z_2+Z_\infty \to \pi^1_{k,3} \to 0,$$
by § 5.1. Also $i^{-1}_{k+1*}(0)$ is generated by $\{t_{k+3,3}\}$, i.e. by $2\{\mathrm{p}h_{k+1,k+1}\}$, which is twice the generator of the infinite summand.

Hence $$\pi^1_{k,3} = Z_2 + Z_2,$$
and is generated by $\{i_{k+1,2}h_{k,k+1}\}$ and $\{i_{k+2,1}\,\mathrm{p}h_{k+1,k+1}\}$. Note that thus $\Delta_*^{-1}(0) = 0$, whence $p_{k+2*}$ is trivial.

(*b*) When $p = 2$, (B) gives

$$\xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+2}) \xrightarrow{\Delta_*} \pi^2_{k,2} \xrightarrow{i_{k+2*}} \pi^2_{k,3} \xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+2}) \to,$$

i.e. $$\to Z_2 \to Z_2 + Z_2 \to \pi^2_{k,3} \to 0,$$
by 5.1 and since $p_{k+2*}$ is trivial by (*a*). Also

$$i^{-1}_{k+2*}(0) = t_{k+3,3*}\,\pi_{k+2}(S^{k+1}),$$

which is generated by

$$\begin{aligned} h^*_{k+1,k+2}\{t_{k+3,3}\} = h^*_{k+1,k+2}\,2\{\mathrm{p}h_{k+1,k+1}\} &= 2h^*_{k+1,k+2}\{\mathrm{p}h_{k+1,k+1}\} \\ &= \{\mathrm{p}h_{k+1,k+1}\}_*\,2\{h_{k+1,k+2}\} = 0. \end{aligned}$$

Thus $$\pi^2_{k,3} = Z_2 + Z_2,$$
and it is generated by $\{i_{k+1,2}h_{k,k+2}\}$ and $\{i_{k+2,1}\,\mathrm{p}h_{k+1,k+2}\}$. Note that $\Delta_*$ is trivial, whence $p_{k+3*}$ is onto.

(*c*) When $p = 3$, (B) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+2}) \xrightarrow{\Delta_*} \pi^3_{k,2} \xrightarrow{i_{k+3*}} \pi^3_{k,3} \xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+2}) \to,$$

i.e. $$\to Z_2 \to \pi^3_{k,2} \to \pi^3_{k,3} \to Z_2 \to 0,$$
since $p_{k+3*}$ is onto by (*b*). Also

$$i^{-1}_{k+3*}(0) = t_{k+3,3*}\,\pi_{k+3}(S^{k+1}),$$

which is generated by

$$\begin{aligned} h^*_{k+1,k+3}\{t_{k+3,3}\} = h^*_{k+1,k+3}\,2\{\mathrm{p}h_{k+1,k+1}\} &= 2h^*_{k+1,k+3}\{\mathrm{p}h_{k+1,k+1}\} \\ &= \{\mathrm{p}h_{k+1,k+1}\}_*\,2\{h_{k+1,k+3}\} = 0. \end{aligned}$$

Hence $i^{-1}_{k+3*}(0) = 0$, and so $\pi^3_{k,3}$ is an extension of $\pi^3_{k,2}$ by $Z_2$. Thus, using the results of § 5.1, we have that

$$\pi^3_{k,3} \textit{ is an extension of } Z_{24} + Z_2 \textit{ by } Z_2 \textit{ for } k \geqslant 8;$$

and $$\pi^3_{4,3} \textit{ is an extension of } Z_\infty + Z_{12} + Z_2 \textit{ by } Z_2.$$

Note that $\Delta_*$ is trivial, whence $p_{k+4*}$ is onto.

This time we make use of Theorem 2.3 (*g*) to calculate the extension. This gives that $\pi^3_{k,3} \approx \pi_{k+3}(P^{k+2}_{k-1})$ since $k \geqslant 4$. It will be shown in the Appendix that $P^{k+2}_{k-1}$ is of the same homotopy type as $S^k \vee P^{k+2}_k$ when $k \equiv 0 \pmod 4$, where $\vee$ denotes attachment at a point. Thus, by Theorem 3 of (9), we have that

$$\pi^3_{k,3} \approx \pi_{k+3}(S^k) + \pi_{k+3}(P^{k+2}_k) \approx \pi_{k+3}(S^k) + \pi^2_{k+1,2}$$

by Theorem 2.3 ($g$). But $\pi^2_{k+1,2} = Z_4$ by § 5.2 ($c$). Hence

$$\pi^3_{k,3} = Z_{24}+Z_4 \quad \textit{for } k \equiv 0 \pmod 4) \textit{ and } \geqslant 8,$$

and is generated by $\{i_{k+1,2}h_{k,k+3}\}$ and $a$ such that

$$p_{k+3,1*}a = \{h_{k+2,k+3}\}, \qquad 2a = \{i_{k+2,1}\,\mathfrak{p}h_{k+1,k+3}\}.$$

Also $$\pi^3_{4,3} = Z_\infty+Z_{12}+Z_4,$$

generated by $\{i_{5,2}\bar{\mathfrak{p}}h_{7,7}\}$, $\{i_{5,2}\mathfrak{E}h_{3,6}\}$, and $a$ such that $p_{7,1*}a = \{h_{6,7}\}$, $2a = \{i_{6,1}\,\mathfrak{p}h_{5,7}\}$.

($d$) When $p = 4$, (B) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+2}) \xrightarrow{\Delta_*} \pi^4_{k,2} \xrightarrow{i_{k+4*}} \pi^4_{k,3} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+2}) \to,$$

i.e. $$\to Z_{24} \to \pi^4_{k,2} \to \pi^4_{k,3} \to Z_2 \to 0$$

since $p_{k+4*}$ is onto by ($c$). Also

$$i^{-1}_{k+4*}(0) = t_{k+3,3*}\pi_{k+4}(S^{k+1}),$$

which is generated by

$$\begin{aligned} h^*_{k+1,k+4}\{t_{k+3,3}\} &= h^*_{k+1,k+4}\,2\{\mathfrak{p}h_{k+1,k+1}\} = 2h^*_{k+1,k+4}\{\mathfrak{p}h_{k+1,k+1}\} \\ &= 2\{\mathfrak{p}h_{k+1,k+1}\}_*\{h_{k+1,k+4}\} = 2\mathfrak{p}_*\{h_{k+1,k+4}\}. \end{aligned}$$

Hence $$i^{-1}_{k+4*}(0) = 2\mathfrak{p}_*\pi_{k+4}(S^{k+1}).$$

Using the results of § 5.1 we thus have that

*$\pi^4_{k,3}$ is an extension of $Z_2$ by $Z_2$ when $k \geqslant 8$, and $\pi^4_{4,3}$ is an extension of $Z_2+Z_2+Z_2$ by $Z_2$.*

To determine the extensions we operate with $h^*_{k+3,k+4}$ on the section of the sequence (B) given in ($c$)† to obtain the diagram, commutative by Lemma 3.1,

$$\begin{array}{ccccccc} \to \pi^4_{k,2} & \xrightarrow{i_{k+4*}} & \pi^4_{k,3} & \xrightarrow{p_{k+3,1*}} & \pi_{k+4}(S^{k+2}) & \to \\ \uparrow{h^*} & & \uparrow{h^*} & & \uparrow{h^*} & \\ \to \pi^3_{k,2} & \xrightarrow{i_{k+3*}} & \pi^3_{k,3} & \xrightarrow{p_{k+3,1*}} & \pi_{k+3}(S^{k+2}) & \to. \end{array}$$

Let $a \in \pi^3_{k,3}$ and $a' \in \pi^4_{k,3}$ be such that

$$p_{k+3,1*}a = \{h_{k+2,k+3}\}, \qquad p_{k+3,1*}a' = \{h_{k+2,k+4}\}.$$

Then

$$p_{k+3,1*}h^*a = h^*p_{k+3,1*}a = h^*_{k+3,k+4}\{h_{k+2,k+3}\} = \{h_{k+2,k+4}\} = p_{k+3,1*}a'.$$

Hence $$h^*a = a'+i_{k+4*}b, \quad \text{where } b \in \pi^4_{k,2}.$$

Accordingly $$2a' = 2h^*a-2i_{k+4*}b = 0$$

since $2h^*a = a_*2\{h_{k+3,k+4}\} = 0$, and since we have just seen that

† I do not use Theorem 2.3 ($g$) again since it no longer holds for $p = k = 4$.

$i_{k+4*}\pi^4_{k,2}$ has only elements of order two. Thus the extension is trivial, and

$$\pi^4_{k,3} = Z_2 + Z_2 \quad \textit{when } k \geqslant 8,$$

generated by $\{i_{k+2,1}\,\mathfrak{p}h_{k+1,k+4}\}$ and $a'$ such that $p_{k+3,1*}a' = \{h_{k+2,k+4}\}$ and

$$\pi^4_{4,3} = Z_2 + Z_2 + Z_2 + Z_2,$$

generated by $\{i_{5,2}\,\bar{p}h_{7,8}\}$, $\{i_{5,2}\,\mathfrak{E}h_{3,7}\}$, $\{i_{6,1}\,\mathfrak{p}h_{5,8}\}$ and $a'$ such that

$$p_{7,1*}a' = \{h_{6,8}\}.$$

Note that in either case

$$\Delta_*\pi_{k+5}(S^{k+2}) = 2\mathfrak{p}_*\pi_{k+4}(S^{k+1}),$$

whence we have by exactness that the image of $p_{k+5*}$ is the $Z_2$ subgroup generated by $12\{h_{k+2,k+5}\}$.

(*e*) When $p = 5$, (B) gives

$$\to \pi_{k+6}(S^{k+2}) \xrightarrow{\Delta_*} \pi^5_{k,2} \xrightarrow{i_{k+5*}} \pi^5_{k,3} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+2}) \to,$$

i.e.
$$\to 0 \to \pi^5_{k,2} \to \pi^5_{k,3} \to Z_2 \to 0$$

since $p_{k+5*}$ is onto the $Z_2$ subgroup of $\pi_{k+5}(S^{k+2})$ by (*d*). Further, by § 5.1, $\pi^5_{k,2} = 0$ when $k \geqslant 8$. Hence

$$\pi^5_{k,3} = Z_2 \quad \textit{when } k \geqslant 8,$$

and is generated by $p^{-1}_{k+3,1*}\,12\{h_{k+2,k+5}\}$.

When $k = 4$, we have from § 5.1 that $\pi^5_{4,2} = Z_2 + Z_2 + Z_2$, whence $\pi^5_{4,3}$ is an extension of $Z_2 + Z_2 + Z_2$ by $Z_2$. We determine this extension as in (*d*) by operating, this time with $h^*_{7,9}$, on the section of the sequence (B) for $k = 4$ given in (*c*). This gives the commutative diagram

$$\begin{array}{ccccccc}
\to \pi^5_{4,2} & \xrightarrow{i_{9*}} & \pi^5_{4,3} & \xrightarrow{p_{7,1*}} & \pi_9(S^6) \to \\
\uparrow h^* & & \uparrow h^* & & \uparrow h^* \\
\to \pi^3_{4,2} & \xrightarrow{i_{7*}} & \pi^3_{4,3} & \xrightarrow{p_{7,1*}} & \pi_7(S^6) \to .
\end{array}$$

Let $a \in \pi^3_{4,3}$ and $\bar{a} \in \pi^5_{4,3}$ be such that $p_{7,1*}a = \{h_{6,7}\}$ and $p_{7,1*}\bar{a} = 12\{h_{6,9}\}$. Then

$$p_{7,1*}h^*a = h^*p_{7,1*}a = h^*_{7,9}\{h_{6,7}\} = 12\{h_{6,9}\} = p_{7,1*}\bar{a}.$$

Hence
$$h^*a = \bar{a} + i_{9*}b, \quad \text{where } b \in \pi^5_{4,2},$$

and
$$2\bar{a} = 2h^*a - 2i_{9*}b = 0$$

since $2h^*a = a_*2\{h_{7,9}\} = 0$ and since $\pi^5_{4,2}$ has only elements of order two. Thus the extension is trivial, and

$$\pi^5_{4,3} = Z_2 + Z_2 + Z_2 + Z_2,$$

generated by $\{i_{5,2}\bar{p}h_{7,9}\}$, $\{i_{5,2}\mathfrak{E}h_{3,8}\}$, $\{i_{6,1}\,\mathrm{p}h_{5,9}\}$, and $\bar{a}$ such that

$$p_{7,1*}\bar{a} = 12\{h_{6,9}\}.$$

7.32. $k \equiv 2 \pmod 4$ *and* $k \geqslant 6$. We first obtain the value of $\{t_{k+3,3}\}$. From § 7.3 we have that

$$\{t_{k+3,3}\} = 2\{\mathrm{p}h_{k+1,k+1}\} - \{g\},$$

where $g\colon S^{k+1} \to i_{k+1,1}(S^k)$ is a symmetric map, $i_{k+1,1}$ being a homeomorphism into. Further, using the notation of §§ 2.3 ($c$) and ($d$), we have from the definition of $g$ that

$$gu_k^{-1} = \phi_{k+2,2}\colon P^k \to i(S^k),$$

which is of degree one (mod 2) and therefore essential. Thus $g$ is essential by Theorem 6.1, i.e. $\{g\} = \{i_{k+1,1}\,h_{k,k+1}\}$.

Hence $\qquad \{t_{k+3,3}\} = 2\{\mathrm{p}h_{k+1,k+1}\} + \{i_{k+1,1}\,h_{k,k+1}\}.$

We are now ready to examine the sequence (B).

($a$) When $p = 1$, (B) gives

$$\xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+2}) \xrightarrow{\Delta_*} \pi^1_{k,2} \xrightarrow{i_{k+1*}} \pi^1_{k,3} \to \pi_{k+1}(S^{k+2}),$$

i.e.
$$\to Z_\infty \to Z_2 + Z_\infty \to \pi^1_{k,3} \to 0,$$

by § 5.1. Also $i^{-1}_{k+1*}(0)$ is generated by $\{t_{k+3,3}\}$: that is, by

$$2\{\mathrm{p}h_{k+1,k+1}\} + \{i_{k+1,1}\,h_{k,k+1}\}.$$

Hence $\qquad \pi^1_{k,3} = Z_4,$

and is generated by $\{i_{k+2,1}\,\mathrm{p}h_{k+1,k+1}\}$. Note that $\Delta_*^{-1}(0) = 0$, whence $p_{k+2*}$ is trivial.

($b$) When $p = 2$, (B) gives

$$\xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+2}) \xrightarrow{\Delta_*} \pi^2_{k,2} \xrightarrow{i_{k+2*}} \pi^2_{k,3} \xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+2}) \to,$$

i.e.
$$\to Z_2 \to Z_2 + Z_2 \to \pi^2_{k,3} \to 0,$$

by § 5.1, and since $p_{k+2*}$ is trivial by ($a$). Also

$$i^{-1}_{k+2*}(0) = t_{k+3,3*}\,\pi_{k+2}(S^{k+1}),$$

which is generated by

$$\begin{aligned} &h^*_{k+1,k+2}(2\{\mathrm{p}h_{k+1,k+1}\} + \{i_{k+1,1}\,h_{k,k+1}\}) \\ &\qquad = \{\mathrm{p}h_{k+1,k+1}\}_*\, 2\{h_{k+1,k+2}\} + \{i_{k+1,1}\,h_{k,k+1}\,h_{k+1,k+2}\} = \{i_{k+1,1}\,h_{k,k+2}\}. \end{aligned}$$

Thus $\qquad \pi^2_{k,3} = Z_2,$

generated by $\{i_{k+2,1}\,\mathrm{p}h_{k+1,k+2}\}$. Note that $p_{k+3*}$ is trivial.

(*c*) When $p = 3$, (B) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+2}) \xrightarrow{\Delta_*} \pi^3_{k,2} \xrightarrow{i_{k+3*}} \pi^3_{k,3} \xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+2}) \to,$$

i.e. $$\to Z_2 \to Z_{24} + Z_2 \to \pi^3_{k,3} \to 0$$

by § 5.1, and since $p_{k+3*}$ is trivial by (*b*). Also

$$i^{-1}_{k+3*}(0) = \{t_{k+3,3}\}_* \pi_{k+3}(S^{k+1}),$$

which is generated by

$$h^*_{k+1,k+3}(2\{\mathfrak{p} h_{k+1,k+1}\} + \{i_{k+1,1} h_{k,k+1}\}) = 12\{i_{k+1,1} h_{k,k+3}\}.$$

Hence $i^{-1}_{k+3*}(0)$ is the $Z_2$ subgroup generated by $12\{i_{k+1,1} h_{k,k+3}\}$.

Thus $$\pi^3_{k,3} = Z_{12} + Z_2,$$

generated by $\{i_{k+1,2} h_{k,k+3}\}$ and $\{i_{k+2,1} \mathfrak{p} h_{k+1,k+3}\}$. Note that again $p_{k+4*}$ is trivial.

(*d*) When $p = 4$, (B) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+2}) \xrightarrow{\Delta_*} \pi^4_{k,2} \xrightarrow{i_{k+4*}} \pi^4_{k,3} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+2}) \to,$$

i.e. $$\to Z_{24} \to Z_{24} \to \pi^4_{k,3} \to 0$$

by § 5.1, and since $p_{k+4*}$ is trivial by (*c*). Also

$$i^{-1}_{k+4*}(0) = \{t_{k+3,3}\}_* \pi_{k+4}(S^{k+1}),$$

which is generated by

$$\begin{aligned} h^*_{k+1,k+4}&(2\{\mathfrak{p} h_{k+1,k+1}\} + \{i_{k+1,1} h_{k,k+1}\} \\ &= 2\{\mathfrak{p} h_{k+1,k+4}\} + \{i_{k+1,1} h_{k,k+1} h_{k+1,k+4}\} = 2\{\mathfrak{p} h_{k+1,k+4}\}. \end{aligned}$$

Thus $$i^{-1}_{k+4*}(0) = 2\mathfrak{p}_* \pi_{k+4}(S^{k+1}),$$

and $$\pi^4_{k,3} = Z_2,$$

generated by $\{i_{k+2,1} \mathfrak{p} h_{k+1,k+4}\}$. Note that, since

$$\Delta_* \pi_{k+5}(S^{k+2}) = 2\mathfrak{p}_* \pi_{k+4}(S^{k+1}),$$

the image of $p_{k+5*}$ is the $Z_2$ subgroup of $\pi_{k+5}(S^{k+2})$.

(*e*) When $p = 5$ and $k \geqslant 10$, (B) gives

$$\to \pi_{k+6}(S^{k+2}) \xrightarrow{\Delta_*} \pi^5_{k,2} \xrightarrow{i_{k+5*}} \pi^5_{k,3} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+2}) \to,$$

i.e. $$\to 0 \to 0 \to \pi^5_{k,3} \to Z_2 \to 0$$

by § 5.1, and since the image of $p_{k+5*}$ is $Z_2$ by (*d*).

Hence $$\pi^5_{k,3} = Z_2 \quad \textit{when } k \geqslant 10,$$

and is generated by $p^{-1}_{k+3,1*} 12\{h_{k+2,k+5}\}$.

(*f*) When $p = 5$ and $k = 6$, (B) gives

$$\to \pi_{12}(S^8) \xrightarrow{\Delta_*} \pi^5_{6,2} \xrightarrow{i_{11*}} \pi^5_{6,3} \xrightarrow{p_{11*}} \pi_{11}(S^8) \to,$$

i.e.
$$\to 0 \to Z_\infty \to \pi^5_{6,3} \to Z_2 \to 0$$

by § 5.1, and since the image of $p_{11*}$ is $Z_2$ by (*d*).

Hence $\pi^5_{6,3}$ *is an extension of* $Z_\infty$ *by* $Z_2$.

To calculate the extension we consider $\pi_{11}(P^8_5)$, $\approx \pi^5_{6,3}$ by Theorem 2.3 (*g*). Let $C^{k+2}$ ($k \geqslant 3$) be the space consisting of an $S^k$ and an $S^{k+1}$ having just one point in common, to which a $k+2$ cell $e^{k+2}$ has been attached by a map $\phi$ such that

$$\phi \mid \dot{E}^{k+2} \to S^k \vee S^{k+1}$$

is essential over $S^k$ and of degree two over $S^{k+1}$. It will be shown in the Appendix that, when $k \equiv 2 \pmod 4$ and $k \geqslant 6$, $(P^{k+2}_{k-1}, P^{k+1}_{k-1})$ is of the same homotopy type as $(C^{k+2}, S^{k+1} \vee S^k)$. So consider the commutative diagram

$$\begin{array}{ccccccccc}
\to \pi_{11}(S^7 \vee S^6) & \xrightarrow{i_{11*}} & \pi_{11}(C^8) & \xrightarrow{j_{11*}} & \pi_{11}(C^8, S^7 \vee S^6) & \xrightarrow{\delta_{11*}} & \pi_{10}(S^7 \vee S^6) \to \\
\uparrow \mathfrak{E} & & \uparrow \mathfrak{E}_1 & & \uparrow \mathfrak{E}_2 & & \uparrow \mathfrak{E} \\
\to \pi_{10}(S^6 \vee S^5) & \xrightarrow{i'_{10*}} & \pi_{10}(C^7) & \xrightarrow{j'_{10*}} & \pi_{10}(C^7, S^6 \vee S^5) & \xrightarrow{\delta'_{10*}} & \pi_9(S^6 \vee S^5) \to .
\end{array}$$

We know that $\pi_{11}(C^8)$ is an extension of $Z_\infty$ by $Z_2$. I shall show that $\pi_{10}(C^7)$ is of finite order and that it contains an element $a$ such that $j_{11*}\mathfrak{E}_1 a$ is non-zero. Thus $\mathfrak{E}_1 a$ is a non-zero element of finite order in $\pi_{11}(C^8)$, which must then be $Z_\infty + Z_2$.

We have by Theorem 3 of (9) that

$$\pi_9(S^6 \vee S^5) \approx \pi_9(S^6) + \pi_9(S^5) = Z_{24} + Z_2,$$

by Theorem 1 of (9) that

$$\pi_{10}(C^7, S^6 \vee S^5) \approx \pi_9(S^6) = Z_{24},$$

and by Theorem 2 of (6) that

$$\pi_{10}(S^6 \vee S^5) \approx \pi_{10}(S^6) + \pi_{10}(S^5) + [\pi_6(S^6), \pi_5(S^5)] = Z_2 + Z_\infty.$$

Thus the lower sequence of the diagram is of the form

$$\to Z_2 + Z_\infty \xrightarrow{i'_{10*}} \pi_{10}(C^7) \xrightarrow{j'_{10*}} Z_{24} \xrightarrow{\delta'_{10*}} Z_{24} + Z_2 \to .$$

Now we have by Theorem 1 of (1) that $i'^{-1}_{10*}(0)$ is the union of

$$\{2h_{6,6} + h_{5,6}\}_* \pi_{10}(S^6) \quad \text{and} \quad [\{2h_{6,6} + h_{5,6}\}_* \pi_6(S^6), \pi_5(S^5)],$$

which, since $\pi_{10}(S^6) = 0$, is generated by

$$[\{2h_{6,6}+h_{5,6}\}_*\{h_{6,6}\}, \{h_{5,5}\}] = 2[\{h_{6,6}\}, \{h_{5,5}\}]+[\{h_{5,6}\}, \{h_{5,5}\}]$$
$$= 2[\{h_{6,6}\}, \{h_{5,5}\}]+\{h_{5,10}\}.$$

Thus $i'_{10*}\pi_{10}(S^6 \vee S^5) = Z_4$, whence $\pi_{10}(C^7)$ is of finite order.

Further, we have by Theorems 1 and 2 of (9) that

$$\delta'_{10*} Z_{24} = \{2h_{6,6}+h_{5,6}\}_* \pi_9(S^6),$$

which is generated by

$$h^*_{6,9}\{2h_{6,6}+h_{5,6}\} = 2\{h_{6,9}\}+h^*_{6,9}\{h_{5,6}\} = 2\{h_{6,9}\}$$

since

$$h^*_{6,9}\{h_{5,6}\} = \{\mathfrak{E}^2 h_{3,4}\, \mathfrak{E}^2 \bar{p} h_{7,7}\} = \mathfrak{E}^2\{h_{3,7}\} = \{\mathfrak{E}^2 h_{3,6}\, \mathfrak{E}^2 h_{6,7}\}$$
$$= 2h_{5,8*}\{h_{8,9}\} = h_{5,8*}\,2\{h_{8,9}\} = 0.$$

Thus the image of $\delta'_{10*} = 2Z_{24}$, whence $j'_{10*}\pi_{10}(C^7)$ is the $Z_2$ subgroup of

$$\pi_{10}(C^7, S^6 \vee S^5).$$

This implies that $\pi_{10}(C^7)$ has eight elements and contains an element $a$, of finite order, such that $j'_{10*}a \neq 0$. But $\mathfrak{E}_2$ is an isomorphism by Theorem 3.2 (B). Hence

$$j_{11*}\mathfrak{E}_1 a = \mathfrak{E}_2 j'_{10*} a \neq 0, \quad \text{whence } \mathfrak{E}_1 a \neq 0.$$

This is what we set out to prove.

Thus $$\pi^5_{6,3} = Z_\infty + Z_2,$$

and is generated by $\{i_{7,2}h_{6,11}\}$ and $a$ such that $p_{9,1*}a = 12\{h_{8,11}\}$ and $2a = 0$.

## Appendix. The homotopy type of certain $P^{k+s}_{k-1}$.†

(A) *The homotopy type of* $P^{k+1}_{k-1}$. Let $\psi_{n+1,m+1}: P^{n-1}_{k-1} \to V_{n,m}$ be the map defined in § 2.3 (*d*). Then $\psi_{k+2,2}: P^k_{k-1} \to S^k$ is a homeomorphism. We also have from § 2.3 (*d*) that

$$g = \psi^{-1}_{k+3,3} t'_{k+2,2}: (E^{k+1}, \dot{E}^{k+1}) \to (P^{k+1}_{k-1}, P^k_{k-1})$$

is a characteristic map for the $(k+1)$-cell in $P^{k+1}_{k-1}$, and, further, that

$$\delta g = \psi^{-1}_{k+2,2} t_{k+2,2}: \dot{E}^{k+1} \to S^k.‡$$

We thus have, by § 2.3 (*b*) (ii), that

$$\{\delta g\} = \begin{cases} \psi^{-1}_{k+2,2*}\,2\{h_{k,k}\} & (k \text{ odd}), \\ 0 & (k \text{ even}). \end{cases}$$

Now let $Y^{k+1}_2$ be the space consisting of an $S^k$ to which one $(k+1)$-cell $e^{k+1}$ has been attached by a map $\phi$ such that

$$\phi \mid \dot{E}^{k+1} \to S^k$$

† $P^{k+s}_{k-1}$ is the projective $k+s$ space $P^{k+s}$ in which a $P^{k-1}$ has been shrunk to a point.

‡ If $f: (E^n, \dot{E}^n) \to (K, L)$, then $\delta f: \dot{E}^n \to L$ is defined by $f \mid \dot{E}^n$.

is of degree two, i.e.

$$\{\delta\phi\} = 2\{h_{k,k}\}, = \psi_{k+2,2*}\{\delta g\}$$

when $k$ is odd. Thus, by Lemma 3 in (**8**), $\psi_{k+2,2}$ can be extended to a homotopy equivalence

$$f_{k+2,2}\colon (P^{k+1}_{k-1}, P^k_{k-1}) \to (Y^{k+1}_2, S^k) \textit{ for all odd } k.$$

Note also that $(Y^{k+2}_2, S^{k+1})$ is of the same homotopy type as $\mathfrak{E}(Y^{k+1}_2, S^k)$.

On the other hand, if $X \vee Y$ denotes the union of two spaces $X$ and $Y$ having just one point in common, then, by the same lemma, we can extend $\psi_{k+2,2}$ to a homotopy equivalence

$$f_{k+2,2}\colon (P^{k+1}_{k-1}, P^k_{k-1}) \to (S^{k+1} \vee S^k, S^k) \textit{ for all even } k.$$

Note again that $(S^{k+2} \vee S^{k+1}, S^{k+1})$ is of the same homotopy type as $\mathfrak{E}(S^{k+1} \vee S^k, S^k)$. Also, by Theorem 3 of (**9**), we have that

$$\pi_{k+1}(S^{k+1} \vee S^k) = i_* \pi_{k+1}(S^k) \dotplus j_* \pi_{k+1}(S^{k+1}),$$

where $i_*$ and $j_*$ are the monomorphisms induced by the identical injections of $S^k$ and $S^{k+1}$. Thus, when $k$ is even, $\pi^1_{k,2}$ and $\pi_{k+1}(S^{k+1} \vee S^k)$ are abstractly isomorphic (see § 5.1), this isomorphism being realized by

$$f_{k+2,2*} \psi^{-1}_{k+3,3*}\colon \pi^1_{k,2} \to \pi_{k+1}(S^{k+1} \vee S^k).$$

Therefore $$f_{k+2,2*} \psi^{-1}_{k+3,3*}\{i_{k+1,1} h_{k,k+1}\} = i_*\{h_{k,k+1}\},$$

$$f_{k+2,2*} \psi^{-1}_{k+3,3*}\{\mathfrak{p} h_{k+1,k+1}\} = j_*\{h_{k+1,k+1}\} + \lambda i_*\{h_{k,k+1}\} \quad (\lambda = 0 \text{ or } 1)$$

provided that the $S^{k+1}$ are suitably oriented.

(B) *The homotopy type of $P^{k+2}_{k-1}$ when $k \equiv 3 \pmod 4$.* We have from § 2.3 ($d$) that

$$g = \psi^{-1}_{k+4,4} t'_{k+3,3}\colon (E^{k+2}, \dot{E}^{k+2}) \to (P^{k+2}_{k-1}, P^{k+1}_{k-1})$$

is a characteristic map for the $k+2$ cell in $P^{k+2}_{k-1}$, and that

$$\delta g = \psi^{-1}_{k+3,3} t_{k+3,3}\colon \dot{E}^{k+2} \to P^{k+1}_{k-1}.$$

But, when $k \equiv 3 \pmod 4$, $\{t_{k+3,3}\} = \{i_{k+1,1} h_{k,k+1}\}$ by § 7.2 ($a$), and $\psi_{k+3,3*}$ is an isomorphism in this dimension by Theorem 2.3 ($g$). So

$$\{\delta g\} = \psi^{-1}_{k+3,3*}\{i_{k+1,1} h_{k,k+1}\} = i'_* \psi^{-1}_{k+2,2*}\{h_{k,k+1}\},$$

where $i'_*$ is induced by the identical injection of $P^k_{k-1} \to P^{k+1}_{k-1}$.

Now let $B^{k+2}_2$ be the space consisting of $Y^{k+2}_2$ to which one $k+2$ cell has been attached by a map $\phi$ such that

$$\phi \mid \dot{E}^{k+2}\colon \dot{E}^{k+2} \to S^k \subset Y^{k+1}_2$$

and is essential: that is

$$\{\delta\phi\} = i_*\{h_{k,k+1}\} = f_{k+2,2*} i'_* \psi^{-1}_{k+2,2*}\{h_{k,k+1}\} = f_{k+2,2*}\{\delta g\},$$

where $i$ is the identical injection and $f_{k+2,2}$ the homotopy equivalence

defined in (A). Hence, by Lemma 3 of (8), we can extend $f_{k+2,2}$ to a homotopy equivalence

$$f'_{k+2,2}\colon (P^{k+2}_{k-1}, P^{k+1}_{k-1}) \to (B^{k+2}_2, Y^{k+1}_2),\ \textit{for}\ k \equiv 3 \pmod 4.$$

Note also that $(B^{k+3}_2, Y^{k+2}_2)$ is of the same homotopy type as $\mathfrak{E}(B^{k+2}, Y^{k+1})$ when $k \geqslant 2$. (When $k = 2$, the attaching map of the 4-cell is to be of Hopf invariant unity.)

(C) *The homotopy type of* $P^{k+2}_{k-1}$ *when* $k \equiv 0 \pmod 4$. We have again from § 2.3 ($d$) that

$$g = \psi^{-1}_{k+4,4} t'_{k+3,3}\colon (E^{k+2}, \dot{E}^{k+2}) \to (P^{k+2}_{k-1}, P^{k+1}_{k-1})$$

is a characteristic map for the $k+2$ cell in $P^{k+2}_{k-1}$, and that

$$\delta g = \psi^{-1}_{k+3,3} t_{k+3,3}\colon \dot{E}^{k+2} \to P^{k+1}_{k-1}.$$

But, when $k \equiv 0 \pmod 4$, $\{t_{k+3,3}\} = 2\{\mathrm{p}h_{k+1,k+1}\}$ by § 7.31. Thus, if $f_{k+2,2*}$ and $j_*$ are as defined in (A), we have that

$$f_{k+2,2*}\{\delta g\} = f_{k+2,2*}\psi^{-1}_{k+3,3*} 2\{\mathrm{p}h_{k+1,k+1}\} = 2j_*\{h_{k+1,k+1}\}.$$

Hence $f_{k+2,2}$ can be extended to a homotopy equivalence

$$f'_{k+2,2}\colon (P^{k+2}_{k-1}, P^{k+1}_{k-1}) \to (Y^{k+2}_2 \vee S^k, S^{k+1} \vee S^k)\ \textit{for}\ k \equiv 0 \pmod 4.$$

But we have from (A) that $Y^{k+2}_2$ is of the same homotopy type as $P^{k+2}_k$. Thus $P^{k+2}_{k-1}$ is of the same homotopy type as $P^{k+2}_k \vee S^k$ when

$$k \equiv 0 \pmod 4.$$

(D) *The homotopy type of* $P^{k+2}_{k-1}$ *when* $k \equiv 2 \pmod 4$, *and* $k \geqslant 6$. When $k \geqslant 3$, let $C^{k+2}$ be the space consisting of $S^{k+1} \vee S^k$ to which one $k+2$ cell has been attached by a map $\phi$ such that

$$\phi \mid \dot{E}^{k+2}\colon \dot{E}^{k+2} \to S^{k+1} \vee S^k$$

is essential over $S^k$ and of degree two over $S^{k+1}$: that is,

$$\{\delta\phi\} = i_*\{h_{k,k+1}\} + 2j_*\{h_{k+1,k+1}\},$$

where $i_*$ and $j_*$ are as defined in (A). If $g$ is the characteristic map for the $k+2$ cell in $P^{k+2}_{k-1}$, we have, by § 2.3 ($d$) as above, that

$$\{\delta g\} = \{\psi^{-1}_{k+3,3} t_{k+3,3}\} \in \pi_{k+1}(P^{k+1}_{k-1}).$$

But, when $k \equiv 2 \pmod 4$,

$$\{t_{k+3,3}\} = 2\{\mathrm{p}h_{k+1,k+1}\} + \{i_{k+1,1} h_{k,k+1}\}$$

by § 7.32. Thus, if $f_{k+2,2}$ is the homotopy equivalence defined in (A), we have that

$$\begin{aligned} f_{k+2,2*}\{\delta g\} &= f_{k+2,2*}\psi^{-1}_{k+3,3*}(2\{\mathrm{p}h_{k+1,k+1}\} + \{i_{k+1,1} h_{k,k+1}\}) \\ &= 2j_*\{h_{k+1,k+1}\} + (2\lambda+1)i_*\{h_{k,k+1}\}, \quad \text{by (A)}, \\ &= \{\delta\phi\}. \end{aligned}$$

Hence, by Lemma 3 in (**8**), we can extend $f_{k+2,2}$ to a homotopy equivalence

$f'_{k+2,2}\colon (P^{k+2}_{k-1}, P^{k+1}_{k-1}) \to (C^{k+2}, S^{k+1}\vee S^k)$ *when* $k \equiv 2 \pmod 4$ *and* $k \geqslant 6$.

Note also that $(C^{k+3}, S^{k+2}\vee S^{k+1})$ is of the same homotopy type as $\mathfrak{E}(C^{k+2}, S^{k+1}\vee S^k)$ when $k \geqslant 3$.

(E) *The homotopy type of* $P^{k+3}_{k-1}$ *when* $k \equiv 2 \pmod 4$ *and* $k \geqslant 6$. When $k \geqslant 3$, let $D^{k+3}_\lambda$ be the space consisting of $C^{k+2}$ to which one $k+3$ cell has been attached by a map $\phi_\lambda$ such that

$$\phi_\lambda \mid \dot{E}^{k+3}\colon \dot{E}^{k+3} \to S^{k+1}\vee S^k \subset C^{k+2},$$

and such that $$\{\delta\phi_\lambda\} = j_*\{h_{k+1,k+2}\}+\lambda i_*\{h_{k,k+2}\}.$$

Again we have from § 2.3 (*d*) that, if $g$ is a characteristic map for the $k+3$ cell in $P^{k+3}_{k-1}$, then

$$\{\delta g\} = \{\psi^{-1}_{k+4,4}\, t_{k+4,4}\} \in \pi_{k+2}(P^{k+2}_{k-1}).$$

But I shall show in § 8.2 of the third paper that, when $k \equiv 2 \pmod 4$ and $k \geqslant 6$, $$\{t_{k+4,4}\} = \{i_{k+2,1}\,\mathrm{p}h_{k+1,k+2}\}.$$

Also, by Theorem 2.3 (*g*), $\psi_{k+4,4*}$ is an isomorphism in this dimension if $k > 2$. Thus we see that

$$\{\delta g\} = \psi^{-1}_{k+4,4*}\, i_{k+2,1*}\{\mathrm{p}h_{k+1,k+2}\} = i'_*\,\psi^{-1}_{k+3,3*}\{\mathrm{p}h_{k+1,k+2}\} \in i'_*\pi_{k+2}(P^{k+1}_{k-1}),$$

where $i'_*$ is induced by the identical injection of $P^{k+1}_{k-1}$ into $P^{k+2}_{k-1}$. If we now consider $\{\delta g\}$ in $\pi_{k+2}(P^{k+1}_{k-1})$ and if $f'_{k+2,2}$ and $f_{k+2,2}$ are the homotopy equivalences defined in (D) and (A), we have that

$$\begin{aligned} f'_{k+2,2*}\{\delta g\} &= f_{k+2,2*}\,\psi^{-1}_{k+3,3*}\{\mathrm{p}h_{k+1,k+2}\} \\ &= h^*_{k+1,k+2}\, f_{k+2,2*}\,\psi_{k+3,3*}\{\mathrm{p}h_{k+1,k+1}\} \\ &= h^*_{k+1,k+2}(j_*\{h_{k+1,k+1}\}+\lambda i_*\{h_{k,k+1}\}) \\ &= j_*\{h_{k+1,k+2}\}+\lambda i_*\{h_{k,k+2}\} \\ &= \{\delta\phi_\lambda\}. \end{aligned}$$

Thus, by Lemma 3 of (**8**), we can extend $f'_{k+2,2}$ to a homotopy equivalence

$\bar{f}_{k+2,2}\colon(P^{k+3}_{k-1}, P^{k+2}_{k-1}) \to (D^{k+3}_\lambda, C^{k+2})$ *for* $k \equiv 2 \pmod 4$ *and* $k \geqslant 6$,

where the $\lambda$ is that determined by $f_{k+2,2*}\,\psi^{-1}_{k+3,3*}\{\mathrm{p}h_{k+1,k+1}\}$. Note also that $(D^{k+4}_\lambda, C^{k+3})$ is of the same homotopy type as $\mathfrak{E}(D^{k+3}_\lambda, C^{k+2})$ when $k > 2$.

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# ON A CONVERGENCE CRITERION OF HARDY AND LITTLEWOOD

*By* M. E. NOBLE (*Nottingham*)

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## 1. Introduction

ONE of a number of results on the convergence of Fourier series due to G. H. Hardy and J. E. Littlewood [(**1**), (**2**)] is as follows:

THEOREM A. *Suppose that $f(x) \in L(-\pi,\pi)$ and that for some $x$*

$$\int_0^t |f(x+u)-f(x)|\,du = o\left\{t\left(\log\frac{1}{|t|}\right)^{-1}\right\} \tag{i}$$

*as $t \to 0$. Suppose also that for this $x$ and for some positive $K$, $\delta$ the Fourier coefficients $a_n$, $b_n$ of $f(x)$ satisfy*

$$a_n \cos nx + b_n \sin nx \geqslant -Kn^{-\delta} \tag{ii}$$

*if $n \geqslant 1$. Then* $$s_n(x) \to f(x)$$
*as $n \to \infty$.*

To avoid repetition I shall assume, whenever $f(x)$ is mentioned, that it is integrable in Lebesgue's sense over $[-\pi,\pi]$ and has a Fourier series

$$\tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx)$$

with partial sums $s_n(x)$.

The proof is curiously indirect involving the method of summability $(V, l)$ [(**3**) 224 or (**4**)], in which

$$s_n \to s(V, l)$$

if, as $n \to \infty$,

$$(2\pi)^{-\frac{1}{2}} n^{\frac{1}{2}l-1} \sum_{m=-n}^{\infty} s_{m+n} \exp(-\tfrac{1}{2}m^2 n^{l-2}) \to s.$$

Hardy and Littlewood showed in (**2**), that under condition (i), $(V, l)$ is 'Fourier-effective' and then used the fact that (ii) is a Tauberian condition for $(V, l)$ if $l = 2(1-\delta)$.

The present paper had its origin in a proof of the following theorem suggested by an earlier (unpublished) result of R. P. Boas, Jr.

THEOREM B. *Suppose that $f(t)$ satisfies condition* (i) *of Theorem A and that there exist a positive number $\delta$ and a sequence of integers $n_k \underset{k}{\to} \infty$ such that $a_n = b_n = 0$ if*

$$|n-n_k| \leqslant n_k^{\delta}.$$

*Then the conclusion of Theorem A holds.*

The proof obtained for Theorem B was direct and in some respects similar to the Hardy–Littlewood proof of the variant of Theorem A in which (i) is replaced by [(1) 252–4]

$$a_n, b_n = O(n^{-\delta}). \tag{ii)$'$}$$

In a subsequent correspondence with me Professor P. B. Kennedy pointed out that the Tauberian argument involved in Theorem A could be used with other Tauberian conditions, in particular with

$$\max |s_m(x)-s_n(x)| = o(1) \quad (|m, n-n_k| \leqslant n_k^{\delta}), \tag{ii)$''$}$$

which includes both (ii)′ and the gap condition of Theorem B (for *some* $\delta$). Further investigation revealed that both the Tauberian argument and the proof found for Theorem B could be adapted to prove the following result, which includes Theorems A and B.

THEOREM 1. *Suppose that $f(t)$ satisfies condition* (i) *of Theorem A and that there exists a positive number $\delta$, and a sequence of integers $n_k \underset{k}{\to} \infty$, such that*

$$\liminf_k [\min\{s_m(x)-s_n(x)\}] \geqslant 0 \quad (|m, n-n_k| \leqslant n_k^{\delta};\ m \geqslant n).$$

*Then*

$$s_{n_k}(x) \to f(x)$$

*as $k \to \infty$.*

In the present paper I shall prove the following more general result, of which Theorem 1 is the most important special case.

THEOREM 2. *Suppose that*

(i) $$\int_0^t |f(x+u)-f(x)|\, du = o\{t\phi(|t|)\}$$

*as $t \to 0$, where $\phi(t) = O\left\{\left(\log\frac{1}{|t|}\right)^{-1}\right\}$ as $t \to 0+$ and is an increasing continuous function of $t$ when $t \geqslant 0$ with $\phi(0) = 0$;*

(ii) *there is a sequence of integers $n_k \underset{k}{\to} \infty$ and real numbers $\lambda_{n_k}$ such that $\lambda_{n_k} = o(n_k)$ and*

$$\int_{n_k^{-1}}^{\lambda_{n_k}^{-1}} u^{-1}\phi(u)\, du = O(1),$$

*and such that*

$$\liminf_k [\min\{s_m(x)-s_n(x)\}] \geqslant 0 \quad (|m, n-n_k| \leqslant \lambda_{n_k};\ m > n).$$

*Then* $$s_{n_k}(x) \to f(x)$$
*as* $k \to \infty$.

Condition (ii) is of course required only if $\int\limits_0 t^{-1}\phi(t)\,dt$ diverges.

A simple *direct* proof of Theorem 2 will be given in § 2; this is perhaps of some interest in that it includes as a particular case a direct proof of Theorem A itself. If condition (i) of Theorem A is weakened by replacing the second member by $O\left\{t\left(\log\frac{1}{|t|}\right)^{-1}\right\}$ we have the theorem:

THEOREM 3. *Suppose that, as* $t \to 0$,

$$\int\limits_0^t |f(x+u)-f(x)|\,du = O\left\{t\left(\log\frac{1}{|t|}\right)^{-1}\right\},$$

*and that there is a sequence of integers* $n_k \to \infty$ *such that, for every* $\delta > 0$,

$$\liminf_k[\min\{s_m(x)-s_n(x)\}] \geqslant 0 \quad (|m,n-n_k| \leqslant n_k^{1-\delta};\ m > n).$$

*Then* $$s_{n_k}(x) \to f(x)$$
*as* $k \to \infty$.

The particular case of the convergence of $s_n(x)$ if $a_n, b_n = O(n^{-1+\delta})$ for every $\delta > 0$ was given by Hardy and Littlewood in (2) [Theorem 7].

In the last section we consider the question as to whether the condition on $\lambda_{n_k}$

$$\int\limits_{n_k^{-1}}^{\lambda_{n_k}^{-1}} \frac{\phi(u)}{u}\,du = O(1)$$

of Theorem 2 can be weakened. This is not easy to decide with $\phi(t)$ satisfying only the conditions of the theorem but it is easy to show that, if $\phi(t)$ satisfies an extra smoothness condition which is trivially satisfied, for instance, by functions of the form $(\log t)^{\alpha_1}\ldots(\log_k t)^{\alpha_k}$, then the condition is best-possible.

## 2. A trigonometric lemma and proof of Theorems 1 and 2

In the proofs of Theorems 1 and 2 we shall assume as usual that $x = 0$ and $f(0) = 0$, and that $f(t)$ is even. We require also the following elementary lemma on trigonometric approximation:

LEMMA. *If* $c_k$ *and* $d_k$ *are sequences of positive real numbers such that* $c_k, d_k \underset{k}{\to} \infty$ *and* $c_k = o(d_k)$, *then, for each* $k$, *we can find a cosine*

*polynomial $P_k(x)$ of order not exceeding $d_k$, with non-negative coefficients, such that*

(a) $$P_k(0) = 1;$$

(b) $$|P_k(x)| \leqslant 2$$

*for $k \geqslant k_0$ uniformly in $x$;*

(c) *if $k \geqslant k_1$ and $c_k^{-1} \leqslant |x| \leqslant \pi$, then*

$$|P_k(x)| \leqslant 8\pi^{-1}c_k d_k^{-1}.$$

We define $g_k(x)$ to have period $2\pi$ and to satisfy

$$g_k(x) = \begin{cases} 1-c_k|x| & (|x| \leqslant c_k^{-1}), \\ 0 & (c_k^{-1} \leqslant |x| \leqslant \pi). \end{cases} \tag{2.1}$$

If $a_m(k)$ is the Fourier cosine-coefficient of $g_k(x)$ of order $m$, an elementary calculation gives, for $m \geqslant 1$,

$$a_m(k) = \frac{2}{\pi}c_k m^{-2}\left\{1-\cos\left(\frac{m}{c_k}\right)\right\} \geqslant 0.$$

Consequently

$$\left|g_k(x) - \sum_0^{[d_k]} a_m(k)\cos mx\right| \leqslant \frac{4c_k}{\pi} \sum_{[d_k]+1}^{\infty} m^{-2}$$

$$\leqslant \frac{6}{\pi}c_k d_k^{-1} \tag{2.2}$$

if $d_k \geqslant 2$, Statement (a) of the lemma can be satisfied by taking

$$P_k(x) = \nu_k \sum_0^{[d_k]} a_m(k)\cos mx$$

for a suitable $\nu_k \underset{k}{\to} 1$; parts (b) and (c) then follow from (2.1), (2.2).

We now proceed to the proof of Theorem 2. We observe that, if the hypotheses of the theorem hold and $p_k$, $q_k$ are two sequences of positive integers such that

$$n_k - \lambda_{n_k} \leqslant q_k \leqslant p_k \leqslant n_k + \lambda_{n_k}, \tag{2.3}$$

$$\liminf_k \frac{p_k - q_k}{\lambda_{n_k}} > 0,$$

and if we write

$$m_k = \tfrac{1}{2}(p_k+q_k), \quad M_k = \tfrac{1}{2}(p_k-q_k),$$

then

$$\int_{m_k^{-1}}^{M_k^{-1}} t^{-1}\phi(t)\,dt = O(1). \tag{2.4}$$

For, if $k$ is sufficiently large, then, for some $\alpha > 1$ (independent of $k$),

$$\int_{m_k^{-1}}^{M_k^{-1}} t^{-1}\phi(t)\,dt \leqslant \int_{\frac{1}{2}n_k^{-1}}^{\alpha\lambda_{n_k}^{-1}} t^{-1}\phi(t)\,dt$$

$$\leqslant O(1)\int_{\frac{1}{2}n_k^{-1}}^{n_k^{-1}} \frac{dt}{t}+\int_{n_k^{-1}}^{\lambda_{n_k}^{-1}} t^{-1}\phi(t)\,dt+O(1)\int_{\lambda_{n_k}^{-1}}^{\alpha\lambda_{n_k}^{-1}}\frac{dt}{t},$$

and all three integrals are obviously bounded as $k \to \infty$.

Suppose now that $M_k$ is an integer and that

$$d_k = M_k,$$

$$c_k = M_k(\log M_k)^{-1}$$

and let $P_k(x)$ be the corresponding polynomial. Then we have, if

$$P_k(x) = \sum_0^{M_k} b_m \cos mx,$$

$$\frac{1}{\pi}\int_0^{\pi} f(t)P_k(t)\frac{\sin(m_k+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

$$= \sum_{p=0}^{M_k} \tfrac{1}{2}b_p \frac{1}{\pi}\int_0^{\pi} \frac{f(t)}{\sin\frac{1}{2}t}\{\sin(m_k+p+\tfrac{1}{2})t+\sin(m_k-p+\tfrac{1}{2})t\}\,dt$$

$$= \sum_{p=0}^{M_k} \tfrac{1}{2}b_p\{s_{m_k+p}(0)-s_{m_k-p}(0)\}. \tag{2.5}$$

We consider the left-hand side of (2.5) and write

$$\frac{1}{\pi}\int_0^{\pi} f(t)P_k(t)\frac{\sin(m_k+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt = \frac{1}{\pi}\left(\int_0^{m_k^{-1}} + \int_{m_k^{-1}}^{\log M_k/M_k} + \int_{\log M_k/M_k}^{c} + \int_c^{\pi}\right)$$

$$\equiv I_1+I_2+I_3+I_4, \tag{2.6}$$

where $c$ is a fixed number such that

$$\int_0^t |f(u)|\,du = O\{t\phi(|t|)\}$$

in $0 \leqslant t \leqslant c$.

In the interval $0 \leqslant t \leqslant m_k^{-1}$,

$$\frac{\sin(m_k+\frac{1}{2})t}{\sin\frac{1}{2}t} = O(m_k),$$

and consequently, by part (*b*) of the lemma,

$$|I_1| \leqslant O\left(m_k \int_0^{m_k^{-1}} |f(t)|\,dt\right) = o\{\phi(m_k^{-1})\} = o(1). \tag{2.7}$$

Write
$$F(u) = \int_0^u |f|\,dt.$$

By part (*b*) of the lemma, for large *k*,

$$|I_2| \leqslant O(1) \int_{m_k^{-1}}^{(\log M_k)M_k^{-1}} u^{-1}|f(u)|\,du$$

$$\leqslant O(1)\left[\left|\left[u^{-1}F(u)\right]_{m_k^{-1}}^{(\log M_k)M_k^{-1}}\right| + \int_{m_k^{-1}}^{(\log M_k)M_k^{-1}} u^{-2}F(u)\,du\right]$$

$$\leqslant o(1)+o(1)\int_{m_k^{-1}}^{M_k^{-1}} u^{-1}\phi(u)\,du+o(1)\int_{M_k^{-1}}^{(\log M_k)M_k^{-1}} \frac{du}{u\log 1/u}, \tag{2.8}$$

$$\leqslant o(1)+o(1)[\log\log u]_{M_k(\log M_k)^{-1}}^{M_k} = o(1)$$

by (2.4). By part (*c*) of the lemma,

$$|I_3| \leqslant \frac{8}{\pi}(\log M_k)^{-1} \int_{(\log M_k)M_k^{-1}}^{c} u^{-1}|f(u)|\,du$$

$$\leqslant \frac{8}{\pi}(\log M_k)^{-1}\left[\left[u^{-1}F(u)\right]_{(\log M_k)M_k^{-1}}^{c} + \int_{(\log M_k)M_k^{-1}}^{c} u^{-2}F(u)\,du\right]$$

$$\leqslant \frac{8}{\pi}(\log M_k)^{-1}\frac{F(c)}{c}+o\big((\log M_k)^{-1}\big)\int_{(\log M_k)M_k^{-1}}^{c} \frac{du}{u\log 1/u} = o(1). \tag{2.9}$$

Finally, also by part (*c*) of the lemma,

$$|I_4| \leqslant O\big((\log M_k)^{-1}\big)\int_c^{\pi} |f(t)|\,dt = o(1). \tag{2.10}$$

Combining (2.7), (2.8), (2.9), and (2.10) we have proved that

$$\sum_{p=0}^{M_k} \tfrac{1}{2}b_p\{s_{m_k-p}(0)+s_{m_k+p}(0)\} = o(1). \tag{2.11}$$

It now follows from (2.3) and the hypothesis of Theorem 2 that, if $\epsilon > 0$ and $k \geqslant k_1(\epsilon)$, then

$$s_{q_k}-\epsilon \leqslant s_n \leqslant s_{p_k}+\epsilon$$

for $q_k \leqslant n \leqslant p_k$. Consequently, by (2.11) and part (*a*) of the lemma,

$$-\epsilon \leqslant s_{p_k}(0)+\epsilon, \quad s_{q_k}(0)-\epsilon \leqslant \epsilon$$

if $k \geqslant k_2(\epsilon) \geqslant k_1(\epsilon)$: that is,

$$\limsup_k s_{q_k}(0) \leqslant 0 \leqslant \liminf s_{p_k}(0). \tag{2.12}$$

If we now make the choices†

$$q_k = n_k, \quad p_k = n_k+\lambda_{n_k}; \qquad q_k = n_k-\lambda_{n_k}, \quad p_k = n_k,$$

then the first choice yields in (2.12)

$$\limsup s_{n_k}(0) \leqslant 0,$$

and the second

$$\liminf s_{n_k}(0) \geqslant 0.$$

Theorem 2 follows.

Theorem 1 is the special case $\phi(t) = (\log 1/t)^{-1}$.

## 3. Proof of Theorem 3

The proof is very similar to that of a special case of Theorem 2, and I merely indicate slight modifications.

Under the hypotheses of Theorem 3, if $\epsilon > 0$, there is a $k_0(\epsilon)$ such that, if $k \geqslant k_0$,

$$\min\{s_m(x)-s_n(x)\} \geqslant -\epsilon \quad (|m,n-n_k| \leqslant n_k^{1-\epsilon};\ m > n).$$

In the lemma of § 2 take

$$d_k = n_k^{1-\epsilon}, \quad c_k = n_k^{1-\epsilon}(\log n_k)^{-1}$$

and let $P_k(x)$ be a corresponding polynomial. Then, arguing as with Theorem 2, we have

$$\sum_0^{[d_k]} \tfrac{1}{2}b_p\{s_{n_k-p}(0)+s_{n_k+p}(0)\} = \frac{1}{\pi}\int_0^\pi f(t)P_k(t)\frac{\sin(n_k+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

$$= \frac{1}{\pi}\left\{\int_0^{n_k^{-1}} + \int_{n_k^{-1}}^{n_k^{\epsilon-1}\log n_k} + \int_{n_k^{\epsilon-1}\log n_k}^{c} + \int_c^\pi\right\}$$

$$= I_1+I_2+I_3+I_4 \text{ (say).}$$

If $n_k^{\epsilon-1}\log n_k \leqslant t \leqslant \pi$ and $k$ is large enough,

$$|P_k(t)| \leqslant \frac{8}{\pi}(\log n_k)^{-1}.$$

Exactly as in the proof of Theorem 2 it follows that

$$|I_1|+|I_2| < \epsilon$$

† If $\lambda_{n_k}$ is odd, replace it by $\lambda_{n_k}-1$, which does not affect hypothesis (ii) of the theorem.

if $k \geqslant k_1(\epsilon)$. Defining $F(u)$ as in § 2 and using part (*b*) of the lemma and condition (i) of Theorem A we have

$$
\begin{aligned}
|I_2| &\leqslant O(1) \int_{n_k^{-1}}^{n_k^{\epsilon-1}\log n_k} u^{-1}|f(u)|\,du \\
&\leqslant O(1)\left[\left|[u^{-1}F(u)]_{n_k^{-1}}^{n_k^{\epsilon-1}\log n_k}\right| + \int_{n_k^{-1}}^{n_k^{\epsilon-1}\log n_k} u^{-2}F(u)\,du\right] \\
&\leqslant O\{(\log n_k)^{-1}\} + O(1) \int_{n_k^{-1}}^{n_k^{\epsilon-1}\log n_k} \frac{du}{u\log 1/u} \\
&\leqslant O\{(\log n_k)^{-1}\} + O(1)\epsilon + o(1) \\
&\leqslant A\epsilon
\end{aligned}
$$

for an absolute constant $A$ if $k \geqslant k_2(\epsilon)$. Finally,

$$
\begin{aligned}
|I_3| &\leqslant \frac{8}{\pi}(\log n_k)^{-1} \int_{n_k^{\epsilon-1}\log n_k}^{c} u^{-1}|f(u)|\,du \\
&\leqslant O\{(\log n_k)^{-1}\}\left\{c^{-1}F(c) + \int_{n_k^{\epsilon-1}\log n_k}^{c} \frac{du}{u\log 1/u}\right\} \\
&= O\left(\frac{\log\log n_k}{\log n_k}\right) < \epsilon
\end{aligned}
$$

if $k \geqslant k_3(\epsilon)$. Consequently, if $k \geqslant \max(k_1, k_2, k_3)$, then

$$\left|\sum_{0}^{[d_k]} \tfrac{1}{2} b_p(s_{n_k-p}(0) + s_{n_k+p}(0))\right| \leqslant (A+2)\epsilon.$$

The proof is completed as for Theorem 2.

## 4. An example

In this section I shall make the further restriction on $\phi(t)$ that there is a fixed positive number $\lambda$ such that, for any fixed positive $k < 1$, as $t \to 0+$

$$\liminf_{t} \frac{\phi(kt)}{\phi(t)} \geqslant \lambda. \tag{4.1}$$

We then have

THEOREM 4. *Suppose that* $\phi(t)$ *satisfies the conditions of Theorem 2 together with* (4.1) *and that* $\lambda_n$ *is a sequence of positive numbers such that* $\lambda_n = o(n)$ *and*

$$\sup\left|\int_{n^{-1}}^{\lambda_n^{-1}} t^{-1}\phi(t)\,dt\right| = \infty. \tag{4.2}$$

*Then there is a continuous periodic function $f(t)$ with Fourier coefficients vanishing when $|n-n_k| \leqslant \lambda_{n_k}$ for a sequence of integers $n_k \underset{k}{\to} \infty$, and such that, as $t \to 0$,*

$$f(t) = o\{\phi(|t|)\}, \tag{4.3}$$

*for which*

$$|s_{n_k}(0)| \to \infty \tag{4.4}$$

*as $k \to \infty$.*

We use a modification of Fejér's well-known construction of a continuous function with a divergent Fourier series and begin with some lemmas.

LEMMA 1. *If $\phi(t)$ is increasing, non-negative, and continuous for $t \geqslant 0$, and*

$$\int_0 t^{-1}\phi(t)\,dt = \infty,$$

*then $\sum p^{-1}\phi(p^{-1})$ and $\sum \phi(p^{-1})$ diverge.*

For, by a trivial change of variable, the divergence of the integral is equivalent to

$$\int^{\infty} t^{-1}\phi(t^{-1})\,dt = \infty,$$

and, since $\phi(t^{-1})$ is decreasing, this is equivalent to the divergence of $\sum p^{-1}\phi(p^{-1})$, which in turn implies that of $\sum \phi(p^{-1})$.

LEMMA 2. *Suppose that $u_n$ is a decreasing sequence of positive numbers, with $\sum u_n$ divergent, such that for some positive number $\lambda$*

$$\liminf_{n} \frac{u_{kn}}{u_n} \geqslant \lambda \tag{4.5}$$

*for any positive integer $k$. Then there are positive numbers $A$, $k_0 > 1$ such that, for every integer $n$, there is a positive integer $m$ with $n \leqslant m \leqslant k_0 n$ such that*

$$\sum_1^m u_p \leqslant Amu_m. \tag{4.6}$$

We write $\sigma_p$ for the $(C, 1)$-means of the sequence $\{u_n\}$ and begin with the identity (easily obtainable by partial summation)

$$\sum_1^N \frac{u_p}{p} = \sum_1^{N-1} \frac{\sigma_p}{p+1} + \sigma_N. \tag{4.7}$$

Suppose now the lemma to be false and let $\delta_k$ be a decreasing sequence with $\delta_k \underset{k}{\to} 0$. Then, for any fixed integer $k$, there are arbitrarily large integers $N$ such that

$$u_m \leqslant \delta_k \sigma_m \tag{4.8}$$

for $N \leqslant m \leqslant kN$. By (4.7),

$$\sum_{N+1}^{kN} \frac{u_p}{p} = \sum_{N}^{kN-1} \frac{\sigma_p}{p+1} + \sigma_{kN} - \sigma_N, \tag{4.9}$$

and by (4.8), since, as is easily verified, $\sigma_p$ decreases,

$$0 \leqslant \sum_{N+1}^{kN} \frac{u_p}{p} \leqslant \delta_k \sum_{N+1}^{kN} \frac{\sigma_p}{p} \leqslant \delta_k \sum_{N}^{kN-1} \frac{\sigma_p}{p+1}. \tag{4.10}$$

Moreover
$$\sum_{N}^{kN-1} \frac{\sigma_p}{p+1} \geqslant \sigma_{kN} \sum_{N}^{kN-1} \frac{1}{p+1} \sim \sigma_{kN} \log k \tag{4.11}$$

as $N \to \infty$. By (4.9), (4.10), and (4.11),†

$$\sigma_N = \{1+\epsilon(k)\} \sum_{N}^{kN-1} \frac{\sigma_p}{p+1}. \tag{4.12}$$

By (4.6), since $u_n$ is a decreasing sequence, if $N \leqslant p \leqslant kN$,

$$\begin{aligned} \sigma_p \geqslant \sigma_{kN} &\geqslant \frac{1}{kN}\left(\sum_{1}^{kn_0} u_p + \tfrac{1}{2}\lambda k \sum_{n_0+1}^{N} u_p\right) \\ &\geqslant \tfrac{1}{2}\lambda\sigma_N + O(N^{-1}). \end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13), we obtain

$$\begin{aligned} \sigma_N &\geqslant \{1+\epsilon(k)\} \sum_{N}^{kN-1} \frac{1}{p+1}\{\tfrac{1}{2}\lambda\sigma_N + O(N^{-1})\} \\ &\geqslant \{1+\epsilon(k)\}\{\tfrac{1}{2}\lambda\sigma_N + O(N^{-1})\}\log k, \end{aligned}$$

which implies, if $k$ is large enough, that

$$\sigma_N = O(N^{-1}),$$

contradicting the divergence of $\sum u_p$.

We now have the following result analogous to a lemma of Hardy and Littlewood [(**2**) Lemma $\alpha$].

LEMMA 3. *There is a positive constant $A'$ such that*

(i) *if $0 \leqslant x \leqslant \frac{1}{2}$, then for any pair of integers $M, N \geqslant 2$*

$$\left|\sum_{M}^{N} \phi(p^{-1}) \frac{\sin px}{p}\right| \leqslant A'\phi(x);$$

(ii) *for all $x$ and any pair of integers $M, N \geqslant 2$,*

$$\left|\sum_{M}^{N} \phi(p^{-1}) \frac{\sin px}{p}\right| \leqslant A'.$$

† I denote by $\epsilon(k)$ a function of $k$, not necessarily the same at each appearance, tending to zero as $k \to \infty$.

Part (ii) follows easily by partial summation from the uniform boundedness of the corresponding sum with the (bounded) factor $\phi(p^{-1})$ omitted. We prove part (i). Suppose that $M \leqslant x^{-1} < N$ and write $\nu = [x^{-1}]+1$. By Lemma 2 applied to $\phi(p^{-1})$ there are constants $A$, $k$ such that, for some $m$ in the range $\nu \leqslant m \leqslant k\nu$, we have

$$\sum_M^\nu \phi(p^{-1}) \leqslant \sum_1^m \phi(p^{-1}) \leqslant Am\phi(m^{-1}).$$

Consequently

$$\left|\sum_M^\nu \phi(p^{-1})\frac{\sin px}{p}\right| \leqslant x\sum_M^\nu \phi(p^{-1}) \leqslant Axm\phi(m^{-1})$$
$$\leqslant Ak\nu x\phi(\nu^{-1}) \leqslant \tfrac{3}{2}Ak\phi(x). \tag{4.14}$$

Also by partial summation

$$\left|\sum_{\nu+1}^N \phi(p^{-1})\frac{\sin px}{p}\right| \leqslant \frac{\phi((\nu+1)^{-1})}{\nu+1}\,|\mathrm{cosec}\,\tfrac{1}{2}x| \leqslant \pi\phi(x). \tag{4.15}$$

Combining (4.14), (4.15), we have

$$\left|\sum_M^N \phi(p^{-1})\frac{\sin px}{p}\right| \leqslant (\tfrac{3}{2}Ak+\pi)\phi(x). \tag{4.16}$$

The same inequality can be obtained more simply if $x \leqslant N^{-1}$ or $M^{-1} < x$.

If now the sequence $\lambda_n$ satisfies (4.2), let $n_k$ be a sequence of integers such that, as $k \to \infty$,

$$k^{-2}\int_{n_k^{-1}}^{\lambda_{n_k}^{-1}} t^{-1}\phi(t)\,dt \to \infty \tag{4.17}$$

and satisfying

$$2n_k \leqslant n_{k+1}, \tag{4.18}$$

and write

$$C_k(t) = \sum_{[\lambda_{n_k}]+1}^{[\frac{1}{2}n_k]} \left\{\frac{\cos(n_k+p)t}{p}\phi(p^{-1}) - \frac{\cos(n_k-p)t}{p}\phi(p^{-1})\right\}$$
$$= -2\sin n_k t \sum_{[\lambda_{n_k}]+1}^{[\frac{1}{2}n_k]} \phi(p^{-1})\frac{\sin pt}{p}.$$

By (4.18) no two of the cosine polynomials $C_k(t)$ involve cosines of the same multiple of $t$. Write

$$f(t) = \sum_1^\infty k^{-2}C_k(t). \tag{4.19}$$

By part (ii) of Lemma 3 the series in (4.19) is uniformly convergent for $-\pi \leqslant t \leqslant \pi$, so that $f(t)$ is continuous and has the series, regarded as a

trigonometric series, as its Fourier series. Moreover the Fourier coefficients of $f(t)$ are zero when $|n-n_k| \leqslant \lambda_{n_k}$ and

$$s_{n_k}(0) = -k^{-2} \sum_{[\lambda_{n_k}]+1}^{[\frac{1}{2}n_k]} p^{-1}\phi(p^{-1})$$

$$= -k^{-2} \int\limits_{\lambda_{n_k}}^{n_k} x^{-1}\phi(x^{-1})\, dx+o(1)$$

$$= -k^{-2} \int\limits_{n_k^{-1}}^{\lambda_{n_k}^{-1}} t^{-1}\phi(t)\, dt+o(1),$$

so that by (4.17), as $k \to \infty$,

$$|s_{n_k}(0)| \to \infty.$$

It remains to show that $f(t)$ satisfies (4.3). If $\epsilon > 0$, choose $k_0(\epsilon)$ such that

$$\sum_{k_0+1}^{\infty} k^{-2} < \epsilon.$$

Then, by Lemma 3 (i), if $|t| \leqslant \frac{1}{2}$,

$$|f(t)| \leqslant 2A'|t|\phi(|t|) \sum_{1}^{k_0} k^{-2}n_k+A'\epsilon\phi(|t|)$$

$$\leqslant (A'+1)\epsilon\phi(|t|) \tag{4.20}$$

if $|t| \leqslant t_1(\epsilon)$, so that (4.3) is fulfilled.

The construction of $f(t)$ depends essentially on the inequality (4.14), which requires the truth of Lemma 2. This last is certainly *false* if $u_p$ is not restricted by *some* such condition as (4.1) and merely satisfies the hypotheses satisfied by $\phi(p^{-1})$ with $\phi(t)$ restricted as in Theorem 2. It is of course possible, or even probable, that Theorem 2 is best-possible with *no* further restriction on $\phi(t)$, but it seems likely that the construction of a suitable counterexample would require other methods.

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# PROPER PINCHERLE BASES IN THE SPACE OF ENTIRE FUNCTIONS

*By* M. G. ARSOVE (*Seattle*)

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## 1. Introduction

THE problem of expanding analytic functions in generalized Taylor series has been dealt with extensively in the literature.† Among the results obtained, some of the most comprehensive are those due to Boas (**3**), derived by means of an extension to Banach spaces of the Paley–Wiener basis approximation theorem [(**11**) 100].

An important special case of the expansion problem occurs when the underlying region is taken as the entire plane (i.e. when the functions in question are entire functions), and it is this case which occupies our attention here. Thus, we shall consider the linear space $\Gamma$ of all entire functions, topologized according to the metric of uniform convergence on compact sets. This space has been studied in a series of papers by V. Ganapathy Iyer (**4**, **5**, **6**, **7**), using the equivalent metric $N(f-g)$, where $N(f)$ is defined in terms of the Taylor coefficients $a_n$ of $f$ as

$$N(f) = \sup\{|a_0|, |a_n|^{1/n}\} \quad (n \geqslant 1).$$

The expansion problem in $\Gamma$ is, in linear-space terminology, just the problem of determining conditions under which a sequence‡ $\{\alpha_n\}$ of functions in $\Gamma$ constitutes a basis for the space. Considerable interest attaches to basis functions of the form

$$\alpha_n(z) = z^n\{1+\lambda_n(z)\}, \tag{1.1}$$

where each $\lambda_n$ is an entire function vanishing at the origin. Since the possibility of expansions in series of such functions was first given by Pincherle (**12**), we shall refer to bases $\{\alpha_n\}$ of this sort as *Pincherle bases*.

A sufficient condition for the sequence $\{\alpha_n\}$ of (1.1) to be a Pincherle basis has been established by Ganapathy Iyer [(**5**) 93–94] in terms of the functions

$$\beta_n(z) = z^n\lambda_n(z), \tag{1.2}$$

the condition being simply that

$$N(\beta_n) \to 0.$$

† A partial bibliography is to be found in Boas (**3**).

‡ Unless stated to the contrary, the sequences with which we deal will be indexed by $n = 0, 1, 2,...$.

I propose here, using results of Boas (**3**) and Narumi (**9**), to show that this condition can be weakened to

$$N(\lambda_n) \to 0$$

and that the Pincherle basis so obtained is, in fact, proper.† It should be remarked, however, that even the weakened condition is not a necessary one, since, if $\phi$ is any entire function vanishing at the origin,

$$\alpha_n(z) = z^n e^{\phi(z)}$$

yields a proper Pincherle basis.

An obvious consequence of our basis theorem is that, if $\{\gamma_n\}$ is any sequence of entire functions uniformly bounded on compact sets, then

$$\alpha_n(z) = z^n\left\{1+\frac{z}{n+1}\gamma_n(z)\right\} \tag{1.3}$$

defines a proper Pincherle basis. In turn, it follows easily from this that such a basis can be constructed from the successive remainders of the exponential function by setting

$$\alpha_n(z) = n!\left\{e^z - \sum_{k=0}^{n-1}\frac{z^k}{k!}\right\}.$$

This suggests a generalization (which I prove by direct methods) for constructing proper Pincherle bases in analogous fashion from a general class of entire functions of exponential type.

Finally, I observe that (1.3) yields proper bases in which each $\alpha_n$ is a polynomial of arbitrarily prescribed degree $\geqslant n$.‡ On the other hand, considerations of linear dependence, which I indicate in § 5, show that the functions $\alpha_n$ in a basis cannot ultimately be polynomials of degree less than $n$.

## 2. Proper bases

The notion of proper basis adopted here is essentially that set forth in (**1**). However, the present approach, along slightly different lines from that of (**1**), seems somewhat preferable.§

Inasmuch as the linear combinations with which we work are to be interpreted as infinite series, the linear-space terminology employed represents a departure from conventional terminology.

† The terminology 'proper basis' is used here in a different sense from that introduced by Ganapathy Iyer [(**6**) 880]. For the present definition see § 2.

‡ With the qualification 'proper' omitted this conclusion follows also from Ganapathy Iyer's theorem.

§ The author is indebted to Professor W. K. Hayman for suggestions leading to this approach.

To make this explicit, we suppose that $\{\alpha_n\}$ is a sequence of functions in $\Gamma$. If

$$\sum_{n=0}^{\infty} c_n \alpha_n = 0$$

implies $c_n = 0$ $(n = 0, 1,...)$ for all sequences $\{c_n\}$ of complex numbers for which the series converges uniformly on compact sets, the sequence $\{\alpha_n\}$ will be called *linearly independent* (otherwise, *linearly dependent*). I shall say that $\{\alpha_n\}$ *spans* a subspace $\Gamma_0$ of $\Gamma$ provided that $\Gamma_0$ consists of all linear combinations

$$\sum_{n=0}^{\infty} c_n \alpha_n,$$

where $\{c_n\}$ is any sequence of complex numbers for which the series converges uniformly on compact sets. A sequence $\{\alpha_n\}$ which is linearly independent and spans a subspace $\Gamma_0$ will be said to be a *basis* in $\Gamma_0$.

It is frequently desirable to require that the series $\sum |c_n \alpha_n|$ converge uniformly on compact sets, and, when this condition is adjoined to those of the preceding definitions, I shall qualify the nomenclature by inserting the word *absolute*. Thus $\{\alpha_n\}$ is *absolutely linearly independent* if $\sum_{n=0}^{\infty} c_n \alpha_n = 0$ implies $c_n = 0$ $(n = 0, 1,...)$ for all sequences $\{c_n\}$ of complex numbers for which $\sum |c_n \alpha_n|$ converges uniformly on compact sets. Similarly, if $\{\alpha_n\}$ is absolutely linearly independent and absolutely spans a subspace $\Gamma_0$ of $\Gamma$, it will be termed an *absolute basis* in $\Gamma_0$.†

Given a sequence $\{\alpha_n\}$ of entire functions, we set

$$M_n(R) = \max|\alpha_n(z)| \quad (|z| = R > 0)$$

and employ the notation of Ganapathy Iyer,

$$|\alpha_n; R| = \sum_{k=0}^{\infty} |A_{nk}| R^k,$$

where

$$\alpha_n(z) = \sum_{k=0}^{\infty} A_{nk} z^k.$$

From the Cauchy inequality

$$|A_{nk}| \leqslant M_n(2R)/(2R)^k$$

we obtain at once

$$M_n(R) \leqslant |\alpha_n; R| \leqslant 2M_n(2R), \tag{2.1}$$

a result which allows the interchange of $M_n(R)$ and $|\alpha_n; R|$ in much of the following work.

It is convenient to introduce a number of conditions applicable to

† This corresponds to the terminology 'absolutely convergent basis' as used by Karlin (8).

sequences $\{\alpha_n\}$ of entire functions. Conditions ($\alpha$) and ($\beta$) are defined by the limiting relationships

$$(\alpha) \qquad \limsup_{n\to\infty}\{M_n(R)\}^{1/n} < +\infty \quad (\text{all } R > 0)$$

and

$$(\beta) \qquad \lim_{R\to\infty}\left\{\liminf_{n\to\infty}\{M_n(R)\}^{1/n}\right\} = +\infty.$$

The further conditions ($A_1$), ($A_2$), ($A_3$), ($B_1$), ($B_2$), ($B_3$) are defined as follows: *for all sequences* $\{c_n\}$ *of complex numbers*

($A_1$) $\quad |c_n|^{1/n} \to 0 \Rightarrow \sum_{n=0}^{\infty}|c_n\alpha_n|$ *converges uniformly on compact sets*;

($A_2$) $\quad |c_n|^{1/n} \to 0 \Rightarrow \sum_{n=0}^{\infty}c_n\alpha_n$ *converges uniformly on compact sets*;

($A_3$) $\quad |c_n|^{1/n} \to 0 \Rightarrow c_n\alpha_n \to 0$ *uniformly on compact sets*;

($B_1$) $\quad \sum_{n=0}^{\infty}|c_n\alpha_n|$ *converges uniformly on compact sets* $\Rightarrow |c_n|^{1/n} \to 0$;

($B_2$) $\quad \sum_{n=0}^{\infty}c_n\alpha_n$ *converges uniformly on compact sets* $\Rightarrow |c_n|^{1/n} \to 0$;

($B_3$) $\quad c_n\alpha_n \to 0$ *uniformly on compact sets* $\Rightarrow |c_n|^{1/n} \to 0$.

The elementary arguments used in proving Theorems 2 and 3 of (**1**) show that conditions ($\alpha$), ($A_1$), ($A_2$), ($A_3$) are equivalent and that conditions ($\beta$), ($B_1$), ($B_2$), ($B_3$) are equivalent. For the sake of completeness I include the proofs here.

THEOREM 1. *For any sequence* $\{\alpha_n\}$ *of entire functions, conditions* ($\alpha$), ($A_1$), ($A_2$), ($A_3$) *are equivalent.*

*Proof.* Suppose that ($\alpha$) holds. Then, given $R > 0$, there corresponds a number $M(R)$ such that $\{M_n(R)\}^{1/n} < M(R)$ for all $n$. Thus, if $|c_n|^{1/n} \to 0$, then

$$\sum_{n=0}^{\infty}|c_n\alpha_n(z)|$$

is dominated on $|z| \leqslant R$ by the convergent series

$$\sum_{n=0}^{\infty}|c_n|\{M(R)\}^n.$$

There results $\qquad (\alpha) \Rightarrow (A_1) \Rightarrow (A_2) \Rightarrow (A_3).$

To complete the proof, I show that

$$(A_3) \Rightarrow (\alpha).$$

For this, let us assume that ($A_3$) holds but ($\alpha$) fails. Choosing $R > 0$,

so that $\{\{M_n(R)\}^{1/n}\}$ is unbounded, we can find an increasing sequence $\{n_k\}$ of positive integers for which

$$\{M_{n_k}(R)\}^{1/n_k} > k.$$

Then the sequence $\{c_n\}$ defined by

$$c_{n_k} = 1/M_{n_k}(R), \qquad c_n = 0 \quad (n \neq n_k)$$

has the property that $|c_n|^{1/n} \to 0$. However, for all $k$,

$$\max|c_{n_k}\alpha_{n_k}(z)| = 1 \quad (|z| = R),$$

and this contradicts $(A_3)$.

Theorem 2. *For any sequence* $\{\alpha_n\}$ *of entire functions, conditions* $(\beta)$, $(B_1)$, $(B_2)$, $(B_3)$ *are equivalent.*

*Proof*. Suppose that $(\beta)$ holds but $(B_3)$ fails. Then, for some sequence $\{c_n\}$ of complex numbers, $c_n\alpha_n \to 0$ uniformly on compact sets and $\limsup_{n\to\infty} |c_n|^{1/n} > \epsilon > 0$. It follows that there exists an increasing sequence $\{n_k\}$ of positive integers such that

$$|c_{n_k}| > \epsilon^{n_k}.$$

Condition $(\beta)$ ensures that, for some $R > 0$,

$$\liminf_{n\to\infty}\{M_n(R)\}^{1/n} > \epsilon^{-1}.$$

Hence, $M_n(R) > \epsilon^{-n}$ for large $n$, so that ultimately

$$|c_{n_k}M_{n_k}(R)| > 1.$$

This contradiction to $c_n\alpha_n \to 0$ uniformly on compact sets yields

$$(\beta) \Rightarrow (B_3) \Rightarrow (B_2) \Rightarrow (B_1),$$

and there remains to establish

$$(B_1) \Rightarrow (\beta).$$

Here we assume that $(B_1)$ holds but $(\beta)$ fails. Then

$$\lim_{R\to\infty}\left[\liminf_{n\to\infty}\{M_n(R)\}^{1/n}\right] < M < +\infty,$$

and the monotonicity of the maximum functions allows us to conclude that, for each $R > 0$,

$$\liminf_{n\to\infty}\{M_n(R)\}^{1/n} < M.$$

There thus exists an increasing sequence $\{n_k\}$ of positive integers such that

$$\{M_{n_k}(R)\}^{1/n_k} < M.$$

Hence, given any $R > 0$, we have

$$|\alpha_{n_k}(z)| < M^{n_k}$$

for $|z| \leqslant R$ and $k > R$. Setting

$$c_{n_k} = (2M)^{-n_k}, \qquad c_n = 0 \quad (n \neq n_k),$$

we then see that $\sum |c_n \alpha_n|$ converges uniformly on compact sets and that $\limsup_{n\to\infty} |c_n|^{1/n} = (2M)^{-1}$. This contradiction completes the proof.

It should be remarked also that Theorems 2 and 3 of (1) show that the condition that $\sum |c_n \alpha_n|$ *converges uniformly on compact sets*, figuring in ($A_1$) and ($B_1$), can be replaced by the condition that $\sum |c_n \alpha_n|$ *converges to a function bounded on compact sets.*

Modifying Ganapathy Iyer's terminology [(4) 880], we state

*Definition* 1. A sequence $\{\alpha_n\}$ of entire functions will be called a *proper basis* for a subspace $\Gamma_0$ of $\Gamma$ provided that

(1) $\{\alpha_n\}$ is a basis in $\Gamma_0$;

(2) for every sequence $\{c_n\}$ of complex numbers the series $\sum c_n \alpha_n$ converges uniformly on compact sets if and only if $|c_n|^{1/n} \to 0$.

The role of Theorems 1 and 2 is now evident: *a basis* $\{\alpha_n\}$ *in a subspace* $\Gamma_0$ *is proper if and only if one of the conditions* ($\alpha$), ($A_1$), ($A_2$), ($A_3$) *holds and one of the conditions* ($\beta$), ($B_1$), ($B_2$), ($B_3$) *holds*. In particular, it is clear that every proper basis is an absolute basis, and that the definition of a proper basis as given here is equivalent to that in (1).

Among the above criteria for a basis to be proper, that consisting of conditions ($\alpha$) and ($\beta$) is of special interest. Further, we note from (2.1) that this criterion remains valid when $M_n(R)$ is replaced by $|\alpha_n; R|$.

As an indication of the importance of proper bases in the study of $\Gamma$, I mention the following interrelationship between proper bases and automorphisms† [see Corollary 4.2 of (1)].

THEOREM 3. *If* $\{\alpha_n^1\}$ *and* $\{\alpha_n^2\}$ *are proper bases in* $\Gamma$, *there exists an automorphism* $T$ *of* $\Gamma$ *such that* $T\alpha_n^1 = \alpha_n^2$ ($n = 0, 1,\dots$). *Conversely, if* $T$ *is an automorphism of* $\Gamma$ *and* $\{\alpha_n^1\}$ *is a proper basis in* $\Gamma$, *then* $\{\alpha_n^2\}$, *where* $\alpha_n^2 = T\alpha_n^1$ ($n = 0, 1,\dots$), *is also a proper basis in* $\Gamma$.

In the sequel we shall be concerned with the problem of determining when a sequence of functions of the form (1.1) yields a basis in $\Gamma$. To prove such a sequence a basis, it suffices to show that it spans $\Gamma$ since functions of this form are obviously linearly independent.‡ Moreover,

† By an *automorphism* of $\Gamma$ is meant a linear homeomorphic mapping of $\Gamma$ onto itself.

‡ If $f = \sum c_n \alpha_n$ converges uniformly on compact sets, then so does the $k$th derived series $f^{(k)} = \sum c_n \alpha_n^{(k)}$. Assuming that $f = 0$ and that $c_k$ is the first non-vanishing coefficient, we obtain the contradiction $k!\,c_k = f^{(k)}(0) = 0$.

if $\{\alpha_n\}$ is a Pincherle basis, then a necessary and sufficient condition for $\{\alpha_n\}$ to be proper is that, for each $R > 0$, we have

$$\limsup_{n\to\infty} |\alpha_n; R|^{1/n} < +\infty.\dagger \tag{2.2}$$

## 3. Extension of the Ganapathy Iyer basis theorem

The central result here is

THEOREM 4. *Let $\{\alpha_n\}$ be a sequence of functions of the form*

$$\alpha_n(z) = z^n\{1+\lambda_n(z)\},$$

*where each $\lambda_n$ is an entire function vanishing at the origin. Then the conditions*

(i) $\lim_{R\to\infty} \left(\limsup_{n\to\infty} |\lambda_n; R|\right) < +\infty$;

(ii) $\lim_{n\to\infty} |\lambda_n; R| = 0$ *for each* $R > 0$;

(iii) $\lambda_n \to 0$ *uniformly on compact sets*;

(iv) $N(\lambda_n) \to 0$

*are mutually equivalent and imply that $\{\alpha_n\}$ is a proper basis in $\Gamma$.*

*Proof.* Since the constant terms in the power-series expansions for the functions $\lambda_n$ all vanish, it is easy to see that

$$|\lambda_n; R| \leqslant t^{-1}|\lambda_n; tR| \quad (t \geqslant 1).$$

This leads at once to equivalence between conditions (i) and (ii). That (ii) and (iii) are equivalent follows from inequalities of the type (2.1) for the functions $\lambda_n$. Finally, (iii) and (iv) are equivalent by virtue of the fact that convergence in the metric of $\Gamma$ is the same as uniform convergence on compact sets.

Turning to the work of Boas (**3**), we invoke his Theorem 4.1 to get the following conclusion. If, for each $R > 0$, there occurs

$$\limsup_{n\to\infty} |\lambda_n; R| < 1, \tag{3.1}$$

then every entire function $f$ can be expanded uniquely as $f = \sum_{n=0}^{\infty} c_n \alpha_n$, the convergence being uniform on compact sets. Thus, conditions (i), (ii), (iii), (iv), which are obviously equivalent to (3.1), imply that $\{\alpha_n\}$ is a basis in $\Gamma$.

† Note, however, that condition (2.2) does not guarantee that a sequence of the form (1.1) spans $\Gamma$. For example, take $\lambda_n(z) = z$ for all $n$. Then (2.2) holds, but all functions in the subspace spanned by $\{\alpha_n\}$ vanish for $z = -1$.

To conclude that $\{\alpha_n\}$ is actually a proper basis, we observe that condition (ii) yields

$$|\alpha_n; R|^{1/n} = R(1+|\lambda_n; R|)^{1/n} \to R \tag{3.2}$$

and then apply (2.2).

That Theorem 4 is indeed stronger than the theorem of Ganapathy Iyer is readily apparent. For example, if

$$\alpha_n(z) = z^n\left(1+\frac{z}{n+1}\right),$$

then $N(\lambda_n) \to 0$ while $N(\beta_n) \to 1$, $\beta_n$ being defined according to (1.2).

An alternative proof of Theorem 4 can be based on a theorem due to Narumi (9), which preceded the work of Boas (3) but is less general. Since this approach is particularly elementary and is substantially simpler in the present case than in that discussed by Narumi, I include the derivation here.

Let $f$ be an entire function having the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

and let us use the coefficients in the expansions

$$\lambda_n(z) = \sum_{k=1}^{\infty} h_{nk} z^k$$

to form the equations

$$a_0 = c_0, \qquad a_n = c_n + \sum_{k=1}^{n} c_{n-k} h_{n-k,k} \quad (n \geqslant 1). \tag{3.3}$$

It is evident that these equations determine $\{c_n\}$ uniquely in terms of $\{a_n\}$ and yield

$$f(z) = c_0 + \sum_{n=1}^{\infty}\left(c_n + \sum_{k=1}^{n} c_{n-k} h_{n-k,k}\right)z^n.$$

By rearrangement there will follow

$$f(z) = \sum_{n=0}^{\infty} c_n \alpha_n(z),$$

the convergence being uniform on compact sets, provided that we can show that the series

$$\sum_{n=1}^{\infty}\left(|c_n| + \sum_{k=1}^{n} |c_{n-k}||h_{n-k,k}|\right)r^n \tag{3.4}$$

converges for all $r > 0$.

To establish the convergence of (3.4), let us suppose that $M$ and $H$ are constants such that

$$|f(z)| \leqslant M, \qquad |\lambda_n(z)| \leqslant H$$

hold for $|z| = R$ and $n = 0, 1, \ldots$. The Cauchy inequalities $|a_n| \leqslant M/R^n$ and $|h_{nk}| \leqslant H/R^k$ in conjunction with (3.3) give rise to the relations

$$|c_0| \leqslant M, \qquad |c_n| \leqslant MR^{-n}+H\sum_{k=1}^{n}|c_{n-k}|R^{-k} \quad (n \geqslant 1).$$

Defining $\{d_n\}$ by

$$d_0 = M, \qquad d_n = MR^{-n}+H\sum_{k=1}^{n}d_{n-k}R^{-k} \quad (n \geqslant 1),$$

we find that $$d_n - R^{-1}d_{n-1} = HR^{-1}d_{n-1},$$

i.e. that $$d_n = R^{-1}(H+1)d_{n-1}.$$

This results in $$d_n = M\left(\frac{1+H}{R}\right)^n,$$

so that $$|c_n| \leqslant M\left(\frac{1+H}{R}\right)^n$$

and $$\sum_{k=1}^{n}|c_{n-k}||h_{n-k,k}| \leqslant M\left(\frac{1+H}{R}\right)^n.$$

In the light of these estimates it is clear that (3.4) is dominated by the series

$$\sum_{n=1}^{\infty} 2M\left(\frac{1+H}{R}\right)^n r^n.$$

The desired convergence is therefore at hand when

$$r < \frac{R}{1+H}. \tag{3.5}$$

Now, by hypothesis (iii) of Theorem 4, it follows that to each $R > 0$ there corresponds an index $m$ such that $|z| = R$ and $n > m$ imply $|\lambda_n(z)| \leqslant 1$. Hence, (3.5) (with $H = 1$) is applicable to the entire function $f - \sum_{n=0}^{m} c_n \alpha_n$, establishing the convergence of (3.4) for all $r > 0$. This completes the proof that $\{\alpha_n\}$ is a basis in $\Gamma$, and the limiting behaviour noted previously in (3.2) shows that $\{\alpha_n\}$ is proper.

An interesting facet of the above derivation is that it furnishes at once a type of Cauchy inequality for expansions in the basis $\{\alpha_n\}$.

THEOREM 5. *Let $\{\alpha_n\}$ be the proper Pincherle basis of Theorem 4 and let $f$ be an entire function having the expansion*

$$f(z) = \sum_{n=0}^{\infty} c_n \alpha_n(z).$$

*If $M$ and $H$ are constants such that*

$$|f(z)| \leqslant M, \qquad |\lambda_n(z)| \leqslant H$$

*hold for $|z| = R$ and $n = 0, 1,...,$ then for all $n$ the coefficients $c_n$ satisfy the inequality*

$$|c_n| \leqslant M\left(\frac{1+H}{R}\right)^n.$$

As noted in § 1, any sequence $\{\gamma_n\}$ of entire functions uniformly bounded on compact sets gives rise to a proper Pincherle basis of the form

$$\alpha_n(z) = z^n\left\{1+\frac{z}{n+1}\gamma_n(z)\right\}.$$

This direct consequence of Theorem 4 can be applied to show that the functions defined by normalizing the successive remainders of the exponential function, i.e.

$$\alpha_n(z) = n!\left\{e^z - \sum_{k=0}^{n-1} \frac{z^k}{k!}\right\}, \tag{3.6}$$

comprise such a basis. In fact, a simple calculation here yields

$$|\gamma_n(z)| \leqslant e^{|z|}.$$

We proceed now to generalize the basis construction given by (3.6).

## 4. Functions of exponential type and proper Pincherle bases

A general method for constructing proper Pincherle bases from entire functions of exponential type is contained in the following result:

THEOREM 6. *Let $\phi$ be an entire function of order 1 and of exponential type, having the power-series expansion*

$$\phi(z) = \sum_{n=0}^{\infty} t_n z^n.$$

*If $t_0 \neq 0$ and $\{n|t_n|^{1/n}\}$ $(n \geqslant 1)$ has a positive lower bound (or, equivalently, if $\phi(0) \neq 0$ and $\{|\phi^{(n)}(0)|^{1/n}\}$ $(n \geqslant 1)$ has a positive lower bound), then the sequence $\{\alpha_n\}$ defined by*

$$\alpha_n(z) = \frac{1}{t_n}\left[\phi(z) - \sum_{k=0}^{n-1} t_k z^k\right]$$

*is a proper Pincherle basis in $\Gamma$.*

*Proof.* Use will be made of the auxiliary function

$$\Phi(z) = \sum_{n=0}^{\infty} |t_n| z^n,$$

which clearly has order 1 and is of exponential type. According to the usual formula for the remainder in a Taylor expansion, we have

$$|\alpha_n; R| = \frac{1}{|t_n|} \sum_{k=n}^{\infty} |t_k| R^k = \frac{\Phi^{(n)}(R_n)}{|n!\, t_n|} R^n$$

for some $R_n$ between 0 and $R$. Since $\Phi^{(n)}(R)$ is non-decreasing for $R > 0$, and since, by hypothesis, $|n!\, t_n|^{1/n} \geqslant \epsilon > 0$, there follows

$$|\alpha_n; R|^{1/n} \leqslant R\epsilon^{-1} |\Phi^{(n)}(R)|^{1/n} \quad (n \geqslant 1).$$

If $\tau$ denotes the type of $\Phi$, then [(2) 11 (2.2.12)]

$$\limsup_{n\to\infty} |\Phi^{(n)}(R)|^{1/n} = \tau.$$

Hence
$$\limsup_{n\to\infty} |\alpha_n; R|^{1/n} \leqslant R\tau\epsilon^{-1}. \tag{4.1}$$

In view of the concluding remarks of § 2 it is evident that all that remains to show is that $\{\alpha_n\}$ spans $\Gamma$. To this end we note first of all that

$$z^n = \alpha_n(z) - \frac{t_{n+1}}{t_n} \alpha_{n+1}(z).$$

Thus, an arbitrary entire function $f$ can be expressed as

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n \left\{\alpha_n(z) - \frac{t_{n+1}}{t_n} \alpha_{n+1}(z)\right\}.$$

From Theorem 1, inequalities (2.1) and (4.1), and the fact that $|a_n|^{1/n} \to 0$, we see that the series $\sum |a_n \alpha_n|$ and $\sum |a_n(t_{n+1}/t_n)\alpha_{n+1}|$ converge uniformly on compact sets. It is therefore permissible to regroup terms to get

$$f(z) = a_0 \alpha_0(z) + \sum_{n=1}^{\infty} \left(a_n - \frac{t_n}{t_{n-1}} a_{n-1}\right) \alpha_n(z), \tag{4.2}$$

the convergence being uniform on compact sets. This completes the proof.

We observe further that (4.2) yields the coefficients of any entire function $f$ in the expansion

$$f = \sum_{n=0}^{\infty} c_n \alpha_n \tag{4.3}$$

as
$$c_0 = a_0, \qquad c_n = a_n - \frac{t_n}{t_{n-1}} a_{n-1} \quad (n \geqslant 1)$$

in terms of the Taylor coefficients $a_n$ of $f$.

Taking $\{\delta_n\}$ as the fundamental basis

$$\delta_n(z) = z^n,$$

we examine the correspondence between the functions $f$ of (4.3) and the functions

$$g = \sum_{n=0}^{\infty} c_n \delta_n = a_0 \delta_0 + \sum_{n=1}^{\infty} \left(a_n - \frac{t_n}{t_{n-1}} a_{n-1}\right) \delta_n.$$

In particular, this correspondence gives rise to the expansion

$$g = \sum_{n=0}^{\infty} a_n \beta_n, \tag{4.4}$$

where
$$\beta_n = \delta_n - \frac{t_{n+1}}{t_n} \delta_{n+1}.$$

From estimates similar to those employed in the proof of Theorem 6 it is not difficult to show that $\{\beta_n\}$ is a proper basis. However, this fact also follows directly from Theorem 3. For let $T$ be an automorphism of $\Gamma$ such that

$$T\delta_n = \alpha_n \quad (n = 0, 1, \ldots).$$

Then
$$Tg = T\left(\sum_{n=0}^{\infty} c_n \delta_n\right) = \sum_{n=0}^{\infty} c_n \alpha_n = f,$$

so that
$$g = T^{-1}f = T^{-1}\left(\sum_{n=0}^{\infty} a_n \delta_n\right) = \sum_{n=0}^{\infty} a_n T^{-1}\delta_n. \tag{4.5}$$

In view of the arbitrariness of $f$ we see by comparing (4.4) and (4.5) that

$$T^{-1}\delta_n = \beta_n \quad (n = 0, 1, \ldots).$$

Hence, $\{\beta_n\}$ is a proper basis in $\Gamma$, and we have proved the theorem:

THEOREM 7. *Let $\phi$ be an entire function of order* 1 *and of exponential type, having the power-series expansion*

$$\phi(z) = \sum_{n=0}^{\infty} t_n z^n.$$

*If $t_0 \neq 0$ and $\{n|t_n|^{1/n}\}$ $(n \geqslant 1)$ has a positive lower bound (or, equivalently, if $\phi(0) \neq 0$ and $\{|\phi^{(n)}(0)|^{1/n}\}$ $(n \geqslant 1)$ has a positive lower bound), then the sequence $\{\beta_n\}$ defined by*

$$\beta_n(z) = z^n\left(1 - \frac{t_{n+1}}{t_n} z\right)$$

*is a proper Pincherle basis in $\Gamma$.*

When $\phi$ is the exponential function, the results of Theorems 6 and 7 appear as special cases of Theorem 4. That this situation is not typical,

however, can be seen by taking for $\phi(z)$ the series formed by selecting alternate terms of $e^z$ and $e^{2z}$:

$$\phi(z) = 1+(2z)+\frac{z^2}{2!}+\frac{(2z)^3}{3!}+\dots.$$

Then for even values of $n$ the leading coefficient in $\lambda_n(z)$ has the numerical value $2^{n+1}/(n+1)$, so that

$$\limsup_{n\to\infty} N(\lambda_n) = +\infty.$$

## 5. Some remarks on polynomial bases

In the concluding paragraph of § 1 it was noted that (1.3) furnishes proper bases $\{\alpha_n\}$ in which each $\alpha_n$ is a polynomial of arbitrarily prescribed degree not less than $n$. I present here a simple linear-dependence argument which shows that the functions $\alpha_n$ in any basis cannot ultimately be polynomials of degree less than $n$.

Suppose that $\{\alpha_n\}$ is a basis for a closed subspace $\Gamma_0$ of $\Gamma$. Indicating explicitly the dependence of the coefficients on the function, we can then write

$$f = \sum_{n=0}^{\infty} c_n(f)\alpha_n$$

for any $f$ in $\Gamma_0$. The coefficients $c_n$ are obviously linear functionals on $\Gamma_0$. That they are also continuous has been shown by Ganapathy Iyer [(5) 92]† for the case of $\Gamma_0 = \Gamma$, and it is readily verified that this proof carries over to arbitrary closed subspaces $\Gamma_0$.

Continuity of the functionals $c_n$ leads at once to

LEMMA 1. *If $\{\alpha_n\}$ is a basis for a closed subspace $\Gamma_0$ of $\Gamma$, then every subsequence of $\{\alpha_n\}$ is a basis for some closed subspace of $\Gamma_0$.*

*Proof.* Let $\{\alpha_{n_k}\}$ $(k = 0, 1,\dots)$ be a subsequence of $\{\alpha_n\}$. This subsequence is clearly a basis for some subspace $\Gamma_1$ of $\Gamma_0$, and it remains to show that $\Gamma_1$ is closed. Thus, let $\{f_j\}$ be a sequence of points of $\Gamma_1$ converging to some $f \in \Gamma$. Then

$$f_j = \sum_{n=0}^{\infty} c_{jn}\,\alpha_n,$$

where $c_{jn} = 0$ for $n \neq n_k$. Since the hypothesis of $\Gamma_0$ closed ensures that $f$ is in $\Gamma_0$, we have

$$f = \sum_{n=0}^{\infty} c_n\,\alpha_n,$$

and the assumed convergence of $\{f_j\}$ to $f$ results in

$$\lim_{j\to\infty} c_{jn} = c_n.$$

† See also Newns (10) 431–2.

Hence $c_n = 0$ for $n \neq n_k$, proving that $f$ lies in $\Gamma_1$ and thereby that $\Gamma_1$ is closed.

This lemma is applied, in turn, to prove

LEMMA 2. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be bases for a closed subspace $\Gamma_0$ of $\Gamma$. Then to each integer $p \geqslant 0$ there correspond functions $\beta_{n_0}, \beta_{n_1},..., \beta_{n_p}$ in $\{\beta_n\}$ such that*

$$\beta_{n_0}, \beta_{n_1},..., \beta_{n_p}, \alpha_{p+1}, \alpha_{p+2},...$$

*is a basis in $\Gamma_0$.*

*Proof.* First of all, not every $\beta_n$ belongs to the subspace $A_0$ spanned by $\alpha_1, \alpha_2,...$ since this subspace is closed and omits $\alpha_0$. Suppose that $\beta_{n_0}$ is not in $A_0$. Then the sequence

$$\beta_{n_0}, \alpha_1, \alpha_2,... \tag{5.1}$$

is linearly independent, and

$$\beta_{n_0} = \sum_{n=0}^{\infty} b_n \alpha_n$$

with $b_0 \neq 0$ and the convergence being uniform on compact sets. Thus $\alpha_0$ is linearly dependent on the functions (5.1), so that these functions span $\Gamma_0$ and thereby comprise a basis in $\Gamma_0$. Repeating this argument with $\alpha_1, \alpha_2,..., \alpha_p$ in turn completes the proof.

It should be noted that Lemmas 1 and 2 remain in force for absolute bases, but we make no use of this fact.

THEOREM 8. *Let $\{\alpha_n\}$ be a basis in $\Gamma$. If the functions $\alpha_n$ are ultimately polynomials, then for infinitely many values of $n$ the polynomial $\alpha_n$ has degree at least $n$.*†

*Proof.* Assuming the assertion false, we have an index $p$ such that, for $n > p$, each $\alpha_n$ is a polynomial of degree not exceeding $n-1$. We then invoke Lemma 2 to replace $\alpha_0, \alpha_1,..., \alpha_p$ by terms of $\{\delta_n\}$ in such a way that

$$\delta_{n_0}, \delta_{n_1},..., \delta_{n_p}, \alpha_{p+1}, \alpha_{p+2},...$$

is a basis in $\Gamma$. There follows

$$m = \max(n_0, n_1,..., n_p) \geqslant p,$$

and the functions $\quad \delta_{n_0}, \delta_{n_1},..., \delta_{n_p}, \alpha_{p+1},..., \alpha_{m+1}$

consist of $m+2$ polynomials of degree not exceeding $m$. They are therefore linearly dependent, and this contradiction completes the proof.

† As is clear from the proof, the theorem remains valid when $\Gamma$ is replaced by any subspace spanned by a sub-sequence of $\{\delta_n\}$.

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# HOMOGENEITY AND ISOTROPY IN GEODESIC SPACES

*By* F. RHODES (*Princeton*)

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## 1. Introduction

In his book [(**1**) V.2] ***Metric methods in Finsler spaces and in the foundations of geometry***, Busemann characterizes Euclidean, hyperbolic, and elliptic spaces as ***geodesic spaces*** with unique geodesics in which triples of points of small enough diameter can be ***moved freely***, and notes that the last condition is fulfilled if the space is ***isotropic*** at any two points. In this paper a direct proof is given that the upper bound of the diameter of triples which can be moved freely in such a space is two-thirds of the diameter of the space; and this result is extended to freely movable $n$-tuples of points of any geodesic space. For the larger class of spaces which are isotropic at one point, the corresponding results on the upper bound of the radius of sets of points which can be rotated freely about the point of isotropy are given.

## 2. Geodesic spaces

The point $z$ of a metric space $E$ (with metric $\rho$) is said to *lie between* the points $x$ and $y$ of $E$ if it is distinct from them and if

$$\rho(x,y) = \rho(x,z)+\rho(z,y).$$

In symbols we write $(xzy)$ when $z$ lies between $x$ and $y$.

Busemann (**2**) has called a metric space $E$ a *G-space* if (i) it is finitely compact, i.e. a bounded infinite set has at least one accumulation point; (ii) given two distinct points $x$, $y$, there is a third point $z$ such that $(xzy)$; (iii) to every point $p$ of the space there corresponds a positive number $\rho_p$ such that, for any two distinct points $x$, $y$ within the solid sphere $S(p,\rho_p)$ having centre $p$ and radius $\rho_p$, a point $z$ with $(xyz)$ exists; (iv) whenever $(xyz_1)$, $(xyz_2)$, and $\rho(y,z_1) = \rho(y,z_2)$ then $z_1 = z_2$.

Every pair $x$, $y$ of points of a $G$-space is joined by a *segment* (an isometric image of a segment of the Euclidean line) $T(x,y)$, which is unique if $x$ and $y$ belong to $S(p,\rho_p)$ for some point $p$; and every segment is contained in a unique *geodesic* (a unique locally isometric image of the whole Euclidean line). A $G$-space in which every pair of points lies on a unique geodesic is called an *S.L. space*. A one-dimensional $G$-space is

congruent to either the Euclidean line or circle. In the rest of the paper it is therefore assumed that the dimension of the space concerned is at least two.

The diameter of a metric space $E$ is

$$\sup \rho(x, y) \quad (x, y \in E).$$

We require also the *radius of a metric space $E$ about a point $z \in E$*, which is here defined as $\sup \rho(z, x)$ $(x \in E)$.

## 3. Homogeneity and isotropy

A metric space is said to be *homogeneous* if it admits a transitive group of isometries. Every point of a homogeneous metric space can be mapped by an isometry on any other point of the space. We can describe this property by saying that single points can be 'moved freely' in the space. It is well known that all geometrical figures in the Euclidean plane can be moved freely there: in axiomatic characterizations of the Euclidean plane, however, no more than the free mobility of triples of points need be postulated. We distinguish between degrees of homogeneity of spaces by means of the classes of subsets which can be moved freely in the space. A metric space is said to be *$n$-point homogeneous* if every isometry between two sets of $n$ (and between two sets of less than $n$) points of the space can be extended to an isometry of the space on itself, and *locally $n$-point homogeneous* if there exists a positive number $\delta$ such that every isometry between two sets of $n$ (and between two sets of less than $n$) points of the space of diameter less than $\delta$ can be extended to an isometry of the space on itself. Sets of $n$ points of an $n$-point homogeneous space, and small-enough sets of $n$ points of a locally $n$-point homogeneous space, are said to be *freely movable*.

A locally 3-point homogeneous metric space satisfies at every point $z$ the following condition: there is a positive number $\delta$ such that, for every set of four points $u$, $u'$, $v$, $v'$ for which

$$\rho(z, u) = \rho(z, u') < \delta, \quad \rho(z, v) = \rho(z, v') < \delta, \quad \rho(u, v) = \rho(u', v'),$$

there is an isometry of the space on itself which maps $u$, $v$, $z$ on $u'$, $v'$, $z$. A metric space which satisfies this condition at a point $z$ is normally said to be *isotropic* at $z$. Here, however, I shall say that the space is *2-point isotropic* at $z$, and introduce other degrees of isotropy corresponding to other degrees of homogeneity. A metric space will be said to be *$n$-point isotropic* at a point $z$ if there exists a positive number $\delta$ such that every isometry which leaves $z$ fixed and maps a set of $n$ points within the distance $\delta$ of $z$ on an isometric set of points can be extended

to an isometry of the space on itself. I shall call such an isometry a *rotation* about $z$ and shall say that $n$-tuples within $\delta$ of $z$ can be *rotated freely* about $z$.

## 4. Homogeneity and isotropy in geodesic spaces

In a locally $n$-point homogeneous metric space 'small enough' sets of $n$ points can be moved freely, while in a metric space which is $n$-point isotropic at a point $z$ sets of $n$ points 'near enough' to $z$ can be rotated freely about $z$. In a locally homogeneous (isotropic) $G$-space the maximum size of freely movable (rotatable) sets can be given explicitly in terms of the diameter of the space (the radius of the space about the point of isotropy).

The first theorem answers the preliminary question: when are two points of a homogeneous $G$-space near enough to ensure that they are joined by a unique segment? The corresponding question for isotropic $G$-spaces is also answered.

THEOREM 1. (*a*) *If $x, y$ are two points of a locally 2-point homogeneous $G$-space $E$ which are joined by more than one distinct segment, then the distance $\rho(x, y)$ is the diameter of the space; if the space is also unbounded, it has unique geodesics.*

(*b*) *If a point $x$ of a $G$-space $E$ which is 1-point isotropic at $z$ is joined to $z$ by more than one distinct segment, then the distance $\rho(z, x)$ is the radius of the space about $z$.*

In the first case suppose that $T_1(x, y)$ and $T_2(x, y)$ are distinct segments and that there are two points $u, v$ in $E$ such that $\rho(u, v) > \rho(x, y)$. A subsegment $T_1(x, y')$ of $T_1(x, y)$ of length less than $\min(\rho_x, \delta)$ can be mapped by an isometry of the space on itself on a subsegment $T_1(u, v')$ of a segment $T_1(u, v)$. This isometry maps $T_1(x, y)$ into $T_1(u, v)$ and $T_2(x, y)$ into a second segment $T_2(u, v)$ which has a common subsegment with $T_1(u, v)$. But this violates condition (iv) for a $G$-space.

Part (*b*) of the theorem can be proved in a similar way.

The maximum diameter of freely movable pairs of points in a locally 2-point homogeneous $G$-space is the diameter of the space. That is to say:

THEOREM 2 (*a*). *A locally 2-point homogeneous $G$-space is 2-point homogeneous.*

We prove first the corresponding isotropy result.

THEOREM 2 (*b*). *Every point of a $G$-space which is 1-point isotropic at $z$ can be rotated freely about $z$.*

For let $x$, $y$ be two points of the space equidistant from $z$ and choose points $x'$, $y'$ such that $(xx'z)$, $(yy'z)$ and

$$\rho(z,x') = \rho(z,y') < \min(\rho_z, \delta).$$

There is a rotation about $z$ which maps $x'$ on $y'$; this rotation maps $T(z,x')$ on $T(z,y')$, and so $x$ on $y$.

It is clear from the definitions that a locally 2-point homogeneous space is 1-point isotropic at every point. The proof of Theorem 2 $(a)$ is now completed by the following lemma. The corresponding isotropy result is stated, but the proof is omitted since it is similar to the proof of the first part of the lemma.

LEMMA 1. $(a)$ *A G-space is 2-point homogeneous if it is 1-point isotropic at every point.*

$(b)$ *A G-space is 2-point isotropic at $z$ if there exists a positive number $\delta$ such that, for every set of three points $x$, $y$, $u$ for which*

$$\rho(z,x) = \rho(z,y) < \delta, \quad \rho(z,u) < \delta, \quad \rho(u,x) = \rho(u,y),$$

*there is a rotation about $z$ which leaves $u$ fixed and maps $x$ on $y$.*

To prove the first part of the lemma, let $x, y$ and $u$, $v$ be two pairs of points such that $\rho(x,y) = \rho(u,v)$. If $z$ is the midpoint of a segment $T(x,u)$, there are points $x'$, $u'$ of $T(x,u)$ such that $(xx'z)$, $(uu'z)$,

$$\rho(z,x') = \rho(z,u')$$

and the segments $T(z,x')$, $T(z,u')$ are unique. There is a rotation about $z$ which maps $x'$ on $u'$ and so $x$ on $u$. If $w$ is the image of $v$ under this rotation, there is a rotation about $x$ which maps $w$ on $y$. Hence the space is 2-point homogeneous.

We now prove a similar lemma† for local $n$-point homogeneity. Again the corresponding isotropy result is stated without proof.

LEMMA 2. $(a)$ *A G-space is locally n-point homogeneous if there is a positive number $\delta$ such that every isometry between two sets of $n$ points $(x_1, x_2, \ldots, x_n)$, $(x_1', x_2, \ldots, x_n)$ of diameter less than $\delta$ which maps $x_1$ on $x_1'$ and leaves the other $n-1$ points fixed can be extended to an isometry of the space on itself.*

$(b)$ *A G-space is n-point isotropic at $z$ if there is a positive number $\delta$ such that every isometry which leaves $z$ fixed and maps $n$ points $(x_1, x_2, \ldots, x_n)$ all within $\delta$ of $z$ on $n$ points $(x_1', x_2, \ldots, x_n)$ can be extended to an isometry of the space on itself.*

† This was proved for $n = 3$ by Busemann (**1**) as part of the proof of his Theorem V 2.1.

Lemma 1 (*a*) shows that Lemma 2 (*a*) is true when $n = 2$; so we can complete the proof of Lemma 2 (*a*) by induction on $n$. Suppose that the lemma is true for $(n-1)$-point homogeneity. Then, if a $G$-space satisfies the conditions of the lemma, it is certainly locally $(n-1)$-point homogeneous. Thus, for any two isometric sets of $n$ points of diameter less than $\delta$, there is an isometry of the space on itself which maps $n-1$ points of one set on $n-1$ points of the other set. By the conditions of the lemma, there is an isometry which leaves these $n-1$ points fixed and maps the $n$th point of one set on the $n$th point of the other. Hence the space is locally $n$-point homogeneous.

The main theorem of the paper now follows.

THEOREM 3. (*a*) *In a locally 3-point homogeneous G-space of diameter d, triples of points of diameter less than $\frac{2}{3}d$ can be moved freely. An unbounded locally 3-point homogeneous G-space is 3-point homogeneous.*†

(*b*) *If a G-space of radius d about a point z is 2-point isotropic at z, pairs of points within $\frac{1}{3}d$ of z can be rotated freely about z.*

As before I give only the proof of the homogeneity form of the theorem. The modifications required for the proof of the isotropy form of the theorem are slight.

By Theorem 2, the space is 2-point homogeneous, from which it follows that two points less than the distance $d$ apart are joined by a unique segment. Let $y, z, u$ and $y, z, v$ be two isometric triples of points of diameter less than $\frac{2}{3}d$, and let $S$ be a sphere with centre $z$ and radius $\delta$ such that triples of points within and on it can be moved freely. Now let $u', v', y'$ be the unique projections of $u, v, y$ on $S$; i.e. $u'$ is such that $(zu'u)$ and $\rho(z, u') = \delta$, etc.

If $\rho(y', u') = \rho(y', v')$, there is an isometry which leaves $z$ and $y'$ fixed and maps $u'$ on $v'$. This isometry leaves $y$ fixed and maps $u$ on $v$; hence, by Lemma 2, triples of diameter less than $\frac{2}{3}d$ can be moved freely. I show that the assumption $\rho(y', u') \neq \rho(y', v')$ leads to a contradiction, so that indeed $\rho(y', u') = \rho(y', v')$. Suppose that $\rho(y', u') > \rho(y', v')$; and consider the set of points between $u$ and $y$ and the set of their projections on $S$. Every point of $T(u, y)$ is less than the maximum distance $d$ from $z$ and therefore has a unique projection on $S$. Moreover, the condition (iv) for $G$-spaces ensures that these projections are distinct.

When the points are near enough to $u$, their projections are further from $y'$ than is $v'$; and, when they are near enough to $y$, their projections

† Elliptic spaces of diameter greater than 1 are bounded locally 3-point homogeneous S.L. spaces, but are not 3-point homogeneous.

are nearer to $y'$ than is $v'$. It follows from Busemann's theorem on the convergence of geodesics [(1) I.3.6] that the projection of $T(u, y)$ on $S$ is a simple arc. Consequently, there is a point $w$ between $u$ and $y$ whose projection $w'$ on $S$ is the same distance from $y'$ as is $v'$; i.e.

$$\rho(y', w') = \rho(y', v').$$

The points $z, y', v'$ can be mapped on the points $z, y', w'$ by an isometry which maps $v$ on a point $x$, say, such that

$$\rho(z, x) = \rho(z, v), \qquad \rho(y, x) = \rho(y, v)$$

and either $(zxw)$ or $(zwx)$ or $x = w$.

From the original assumption that $y, z, u$ and $y, z, v$ are isometric, it now follows that

$$\rho(z, x) = \rho(z, u), \qquad \rho(y, x) = \rho(y, u).$$

We complete the proof of the theorem by showing that $x$, $w$, $u$ are the same point, and thus that $w' = u'$ and $\rho(y', u') = \rho(y', v')$. This is clear if $x = w$.

Suppose that $(zxw)$. From the triangle inequalities for $zuw$ and $xwy$, we have

$$\rho(z, u)+\rho(u, w) \geqslant \rho(z, w) = \rho(z, x)+\rho(x, w)$$

with equality only if $u$ lies on $T(z, w)$; and

$$\rho(w, y) \geqslant \rho(x, y)-\rho(x, w)$$

with equality only if $w$ lies on $T(x, y)$. Thus

$$\rho(z, x)+\rho(x, y) \leqslant \rho(z, u)+\rho(u, y),$$

with equality only if both these conditions hold. Since the two sides of the inequality are in fact equal, $u$ lies on $T(z, w)$; and, since also $\rho(z, u) = \rho(z, x)$, it follows that $u$ is $x$. Moreover, $w$ lies on $T(x, y)$, and, since the union of $T(z, x)$ and $T(x, y)$ is not a segment (except in the trivial case where $zvy$ and $zuy$ are linear triples), it follows that $x$ is $w$.

Finally, suppose that $(zwx)$. Then

$$\rho(x, y) \leqslant \rho(x, w)+\rho(y, w).$$

However, $$\rho(x, w) = \rho(x, z)-\rho(w, z) = \rho(u, z)-\rho(w, z),$$

and further $$\rho(u, z) \leqslant \rho(u, w)+\rho(w, z).$$

Hence $$\rho(x, y) \leqslant \rho(u, w)+\rho(y, w) = \rho(u, y) = \rho(x, y),$$

with equality only if $w$ lies on $T(x, y)$ and on $T(u, z)$. It follows that $u$, $x$, $w$ are the same point.

It is now easy to prove the $n$-point analogue of Theorem 3.

THEOREM 4. (*a*) *In a locally n-point homogeneous G-space of diameter d, n-tuples of diameter less than* $\frac{3}{4}d$ *can be moved freely. An unbounded locally n-point homogeneous G-space is n-point homogeneous.*

(*b*) *If a G-space of radius d about a point z is n-point isotropic at z, n-tuples within* $\frac{1}{4}d$ *of z can be rotated freely about z.*

We may suppose that $n > 3$. Let $(u, z, x_1, \ldots, x_{n-2})$, $(v, z, x_1, \ldots, x_{n-2})$ be two sets of $n$ points with $n-1$ points in common of diameter less than $\frac{3}{4}d$ such that

$$\rho(u,z) = \rho(v,z), \qquad \rho(u,x_i) = \rho(v,x_i)$$

for every integer $i$ not exceeding $n-2$. Further, let $u'$, $v'$, $x_i'$ be the unique projections of $u$, $v$, $x_i$ on a sphere $S$ with centre $z$ and radius less than $\delta$.

If $\rho(u',x_i') > \rho(v',x_i')$ for some integer $i \leqslant n-2$, then there is a point $w_i$ of $T(u,x_i)$ whose projection $w_i'$ on $S$ satisfies the equation

$$\rho(w_i', x_i') = \rho(v', x_i').$$

We can map $z$, $x_i'$, $v'$ on $z$, $x_i'$, $w_i'$ by an isometry of the space on itself; but this gives rise to an impossible figure as in the proof of Theorem 3.

Thus $\rho(u',x_i') = \rho(v',x_i')$ for every integer $i \leqslant n-2$; and there is an isometry which leaves $z$ and every $x_i'$ fixed and maps $u'$ on $v'$. This isometry leaves every $x_i$ fixed and maps $u$ on $v$. Lemma 2 now completes the proof of the homogeneity form of the theorem; the isotropy form is proved in almost the same way.

A locally $n$-point homogeneous metric space is clearly $(n-1)$-point isotropic at every point, and it follows from Lemma 2 that a $G$-space which is $(n-1)$-point isotropic at every point is locally $n$-point homogeneous. However, it is easy to see that a $G$-space which is isotropic at two non-conjugate points is locally $n$-point homogeneous.

THEOREM 5. *A G-space of diameter d which is $(n-1)$-point isotropic at two points less than the distance d apart is locally n-point homogeneous. If the space has unique geodesics and is $(n-1)$-point isotropic at any two points, it is locally n-point homogeneous.*

It is only necessary to show that the space is 1-point homogeneous, and this is contained in the following proof that 1-point isotropy at two points implies 2-point homogeneity.

Suppose the space is 1-point isotropic at the points $x$ and $y$. Let $u \in T(x,y)$, $v \notin T(x,y)$, $w \in T(u,v)$, and let $z$ be the point such that $(zwy)$ and $\rho(z,y) = \rho(x,y)$. If the space has unique geodesics or if $\rho(x,y) < d$, the points $z$ are distinct from each other and from $x$. By Busemann's

theorem on the convergence of geodesics, the set of points $z$, constructed from a fixed segment $T(u,v)$, is a simple arc with $x$ as one end-point. For each point $z$ there is a rotation about $y$ which maps $x$ on $z$, and hence the space is 1-point isotropic at $z$. Since the space is also 1-point isotropic at $x$, it is 1-point isotropic at every point of a sphere with centre $x$. It is easy to see that the set of points of isotropy is both open and closed, and is therefore the whole space; whence the space is 2-point homogeneous.

Finally, we characterize isotropy at a point $z$ in terms of homogeneity of spheres with centre $z$.

THEOREM. 6. *A G-space of radius $d$ about a point $z$ is $n$-point isotropic at $z$ if and only if spheres with centre $z$ and radius less than $\frac{1}{4}d$ are $n$-point homogeneous under the group of rotations about $z$.*

The condition is clearly necessary, and the proof of its sufficiency is contained in the proof of Theorem 4.

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# NOTE ON A STEP FUNCTION

*By* S. M. SHAH *and* S. K. SINGH (*Aligarh*)

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## 1. Introduction

In this note we prove a relation satisfied by a step function with 'unit jumps' and apply it in the theory of entire and meromorphic functions.

Theorem. *Let $f(x)$ be a step function such that*

$$f(x) = \sum_{r_n \leqslant x} 1,$$

*where* $r_n$ $(0 < r_1 \leqslant r_2 \leqslant \ldots;\ r_n \to \infty$ *with* $n)$

*are jump points*† *of $f(x)$; and let*

$$I(x) = \int_1^x \frac{f(t)}{t}\,dt,$$

$$\limsup_{x\to\infty} \frac{\log f(x)}{\log x} = B, \qquad \liminf_{x\to\infty} \frac{\log f(x)}{\log x} = A,$$

*where* $0 < B \leqslant \infty, \quad 0 \leqslant A < \infty.$

*Then*
$$J(f) = \limsup_{x\to\infty} \frac{I(x)}{f(x)\log x} \leqslant 1 - \frac{A}{B}. \tag{1}$$

Corollary 1. *We have* [(2) Theorem 3], *for* $0 \leqslant A < \infty, 0 < B \leqslant \infty,$

$$1\Big/\liminf_{x\to\infty} \frac{\log I(x)}{\log\log x} \leqslant J(f) \leqslant 1 - \frac{A}{B}. \tag{2}$$

It is easily seen that

$$H(f) = \liminf_{x\to\infty} \frac{\log f(x)}{\log\log x} < \infty$$

if and only if $G(f) = \liminf_{x\to\infty} \dfrac{\log I(x)}{\log\log x} < \infty$. Hence the left-hand inequality of (2) may be useful only when $G(f) < \infty$ and hence when $A = 0$. It is obvious that we may have

$$A = 0, \qquad G(f) = H(f) = \infty.$$

† A jump point is repeated as many times as there are units in the jump.

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COROLLARY 2. *Let*

(i) *$F(x)$ be integrable in any interval* $(1, X)$,

(ii) *$F(x) \sim f(x)$ as $x \to \infty$, where $f(x)$ is a step function defined in the theorem,*

(iii) $$\limsup_{x\to\infty} \frac{\log F(x)}{\log x} = B, \quad \liminf_{x\to\infty} \frac{\log F(x)}{\log x} = A,$$

*where* $0 < B \leqslant \infty$; $0 \leqslant A < \infty$.

*Then* $$J(F) \leqslant 1 - \frac{A}{B}. \tag{3}$$

*Proof of* (1). We may suppose that $r_1 \geqslant 1$. Then†

$$I(x) = \sum_{r_n \leqslant x} \log \frac{x}{r_n} = f(x) \log x - \sum_{r_n \leqslant x} \log r_n. \tag{4}$$

Further $$\limsup_{x\to\infty} \frac{\log f(x)}{\log x} = \limsup_{n\to\infty} \frac{\log n}{\log r_n} = B.$$

Hence $\log n / \log r_n < B + \epsilon$ for all $n > n_0$,

$$\sum_{r_n \leqslant x} \log r_n > O(1) + \frac{1}{B+\epsilon} \sum_{r_n \leqslant x} \log n.$$

If $N$ be the largest integer such that $r_N \leqslant x$, we have

$$\sum_{r_n \leqslant x} \log r_n > \frac{1}{B+\epsilon} \{N \log N + O(N)\} = \frac{1}{B+\epsilon} [f(x) \log f(x)] + O\{f(x)\},$$

and so

$$I(x) \leqslant f(x) \log x - \frac{1}{B+\epsilon} f(x) \log f(x) + O\{f(x)\},$$

$$J(f) \leqslant 1 - \frac{1}{B+\epsilon} \liminf_{x\to\infty} \frac{\log f(x)}{\log x} = 1 - \frac{A}{B+\epsilon},$$

and (1) follows.

*Proof of Corollary* 2. Put

$$F(x) = f(x) + g(x).$$

Then $g(x)$ is integrable in every interval $(1, X)$ and

$$g(x) = o\{f(x)\},$$

$$\int_1^x \frac{F(t)}{t}\, dt = \int_1^x \frac{f(t)}{t}\, dt + o\{F(x) \log x\}.$$

Hence $$J(F) = \limsup_{x\to\infty} \frac{\int_1^x \{F(t)/t\}\, dt}{F(x) \log x} = J(f) \leqslant 1 - \frac{A}{B}.$$

† From (4) we have $J(f) \leqslant 1$ for $0 \leqslant A \leqslant B \leqslant \infty$.

## 2. Applications

Let
$$\phi(z) = \sum_0^\infty a_n z^n$$
be an entire function of order $\rho$ $(0 < \rho < \infty)$ and lower order $\lambda$ and let $\mu(r) = \mu(r, \phi)$ denote the maximum term and $\nu(r)$ its rank. It is known (3), (4) that
$$\limsup_{r\to\infty} \frac{\log \nu(r)}{\log r} = \rho, \qquad \liminf_{r\to\infty} \frac{\log \nu(r)}{\log r} = \lambda,$$
$$\log M(r) \sim \log \mu(r) = c + \int_1^r \frac{\nu(t)}{t}\, dt.$$
Hence from (1) it follows that†
$$\limsup_{r\to\infty} \frac{\log M(r)}{\nu(r)\log r} \leqslant 1 - \frac{\lambda}{\rho}. \tag{5}$$
Similarly, if $n(r, a)$ denotes the number of $a$-points of a meromorphic function $\Phi(z)$ in $|z| \leqslant r$ and if
$$N(r,a) = \int_0^r \frac{n(x,a) - n(0,a)}{x}\, dx + n(0,a)\log r,$$
then
$$\limsup_{r\to\infty} \frac{N(r,a)}{n(r,a)\log r} \leqslant 1 - \frac{\lambda_1(a)}{\rho_1(a)},$$
where
$$\limsup_{r\to\infty} \frac{\log^+ n(r,a)}{\log r} = \rho_1(a), \qquad \liminf_{r\to\infty} \frac{\log^+ n(r,a)}{\log r} = \lambda_1(a),$$
it being assumed that $0 < \rho_1(a) \leqslant \infty$; $0 \leqslant \lambda_1(a) < \infty$.

## 3. Examples

To show that (1) gives the best possible result we construct a step function $f(x, \alpha)$ for which, for one suitable value of $\alpha$, the equality sign holds in (1), and for every other value of $\alpha$ the inequality sign holds in (1). We also give examples to show that, when $A = B = \infty$ or when $A = B = 0$, $J(f)$ may have any assigned value $\alpha$ such that $0 \leqslant \alpha \leqslant 1$.

If $0 < A = B < \infty$, it follows from (1) that $J(f) = 0$. We give an example to show that the converse is not true: that is, we may have $J(f) = 0$, $A \neq B$.

In what follows, we suppose that $(x_n)$ is a sequence of positive numbers increasing sufficiently rapidly.

† R. P. Srivastava has proved (5) by a different method; see *Ganita* 7 (1956) 29–43.

(1)† Let $0 < A < B < \infty$, $1 < \alpha \leqslant B/A$ and let

$$f(x,\alpha) = f(x) = \begin{cases} [x_n^B] & \text{when } x \in I_1 \ (x_n \leqslant x < x_n^\alpha), \\ [x^A x_n^{B-\alpha A}] & \text{when } x \in I_2 \ (x_n^\alpha \leqslant x < e^{x_n}), \\ [x^B] & \text{when } x \in I_3 \ (e^{x_n} \leqslant x < x_{n+1};\ n = 1,2,\ldots). \end{cases}$$

Then
$$\limsup_{x\to\infty} \frac{\log f(x)}{\log x} = B, \qquad \liminf_{x\to\infty} \frac{\log f(x)}{\log x} = A,$$

Further, for $x$ in $I_1$,

$$J(f,x) = \frac{I(x)}{f(x)\log x} = \frac{x_n^B \log(x/x_n) + O(x_n^B)}{x_n^B \log x} \sim 1 - \frac{\log x_n}{\log x}.$$

Hence, for $x$ in $I_1$,

$$J(f,x) \leqslant 1-\alpha^{-1}+o(1),$$
$$J(f,x_n^\alpha) = 1-\alpha^{-1}+o(1).$$

Similarly, for $x$ in $I_2$,

$$J(f,x) \leqslant 1-\alpha^{-1}+o(1).$$

For $x$ in $I_3$,

$$J(f,x) = \left\{\frac{x^B - e^{Bx_n}}{B} + O(x_n^{B-\alpha A} e^{Ax_n})\right\} \Big/ x^B \log x = o(1).$$

Hence
$$J(f) = \limsup_{x\to\infty} J(f,x) = 1-\alpha^{-1}.$$

(2) Let

$$f(x,\alpha,\beta) = f(x) = \begin{cases} [x\,2^x] & \text{when } x_n \leqslant x < X_n = x_{n+1}^\alpha (\log x_{n+1})^\beta, \\ [X_n\, 2^{X_n}] & \text{when } X_n \leqslant x < x_{n+1},\ n = 1,\,2,\ldots, \end{cases}$$

where either $0 < \alpha < 1$, or $\alpha = 0$, $\beta > 1$, or $\alpha = 1$, $\beta < 0$. Then $A = B = \infty$ and $J(f) = 1-\alpha$.

(3) Let

$$f(x,\Delta) = f(x) = \{(\log x)^\Delta \log\log x\} \quad (\Delta \geqslant 0),$$

then
$$A = B = 0, \qquad J(f) = \frac{1}{\Delta+1}.$$

(4) Let
$$f(x) = \{\exp(\log\log x)^2\}.$$

Then $A = B = 0$, $J(f) = 0$ [see (**1**) Theorem 8 b].

† We thank the referee for Examples 1 and 2. The examples that we gave were rather lengthy.

(5) Let $0 \leqslant A < B < \infty$,

$$X_n = \exp\{\log x_n(1+1/\log\log x_n)\},$$

$$C = \frac{B\log x_n}{\log X_n} - A,$$

$$f(x) = \begin{cases} [x_n^B] & (x_n \leqslant x < X_n), \\ [x^A X_n^C] & (X_n \leqslant x < e^{x_n}), \\ [x^B] & (e^{x_n} \leqslant x < x_{n+1}). \end{cases}$$

Then

$$\limsup_{x\to\infty} \frac{\log f(x)}{\log x} = B, \qquad \liminf_{x\to\infty} \frac{\log f(x)}{\log x} = A,$$

$$J(f) = 0.$$

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# ON HADAMARD COMPOSITION WITH ALGEBRAIC-LOGARITHMIC SINGULARITIES

*By* R. WILSON (*Swansea*)

[Received 28 March 1957]

## 1. Introduction

Let $f(z)$, $g(z)$ be two functions represented by the Taylor series

$$f(z) = a_0+a_1 z+a_2 z^2+...+a_n z^n+...,$$
$$g(z) = b_0+b_1 z+b_2 z^2+...+b_n z^n+...,$$

each of finite, non-zero radius of convergence. Then we have [(1) 346–7, (3) 731, (4) 157–9]:

THEOREM A. *The star-domain of the Hadamard composition function of $f(z)$ and $g(z)$,*

$$h(z) = a_0 b_0+a_1 b_1 z+a_2 b_2 z^2+...+a_n b_n z^n+...,$$

*contains the products of the star-domains of $f(z)$ and $g(z)$.*

This does not tell us where the singularities of $h(z)$ *must* lie but only where they *may* lie. However, Pólya has dealt with the case in which $f(z)$ has on its circle of convergence only one singularity, at $z = \alpha$, say, while $g(z)$ may have on its circle of convergence any kind of singular point or singular arc. Let $z = \beta$ be a point on the circle of convergence of $g(z)$ at which it is singular. At $z = \beta$ there may be a unique singularity: that is, one which is isolated on the circumference of the circle of convergence, or a limit-point of such unique singularities, or $z = \beta$ may be an end-point or an interior point of a singular arc of the circle of convergence over which the singularities are dense. The problem is to determine whether the point $\alpha\beta$ is singular for $h(z)$ or not.

This problem has been solved by Pólya in a number of important cases [(3) 766–8 Th. VIII] and, in particular, he obtained the result given in the theorem below:

THEOREM B. *If the sole singularity of $f(z)$ is a pole at the point $z = \alpha$, then the point $\alpha\beta$ is singular for $h(z)$ whatever kind of singularity $g(z)$ may have at the point $z = \beta$.*

Since, in this case,

$$\lim_{n\to\infty} |a_n|^{1/n} = |\alpha|^{-1}, \qquad \limsup_{n\to\infty} |b_n|^{1/n} = |\beta|^{-1},$$

it is clear that $|z| = |\alpha\beta|^{-1}$ is the circle of convergence of $h(z)$, but it is

not obvious that every point $\beta$ of a singular arc of $g(z)$ will give rise to a singular arc at the corresponding points $\alpha\beta$ for $h(z)$. In fact, if we replace the pole at $z = \alpha$ by an essential point, then the result does not necessarily follow. For example, let

$$f(z) = \sum_0^\infty F(n)z^n, \quad \text{where} \quad F(z) = \prod_1^\infty (1-z/n^2),$$

$$g(z) = \sum_0^\infty z^{n^2}.$$

Then $h(z) \equiv 0$. Here $f(z)$ has an isolated essential point at $z = 1$ while $g(z)$ has the circle $|z| = 1$ as a natural boundary.

The object of the paper is to show that the result of Theorem B continues to hold if the pole at $z = \alpha$ is replaced by an algebraic-logarithmic singularity with a single dominant element.

## 2. The algebraic-logarithmic singularity

An algebraic-logarithmic singularity at $z = \alpha$ consists of a finite number of elements of the form

$$(z-\alpha)^{-s_\nu}[\log\{(z-\alpha)^{-1}\}]^{k_\nu}\psi_\nu(z), \tag{1}$$

where $\operatorname{re} s_\nu = \sigma_\nu$, $k_\nu$ is a non-negative integer, and $\psi_\nu(z)$ is regular and non-zero at $z = \alpha$, the combination of $k_\nu = 0$ and $s_\nu$ a negative integer being excluded [(**2**) 274]. If $s_\nu \neq 0, -1, -2,\ldots$, the number pair $[\sigma_\nu, k_\nu]$ is called the *weight* of the singular element (1), while, if $s_\nu = 0, -1, -2,\ldots$, $k_\nu > 0$, the number pair $[s_\nu, k_\nu - 1]$ is defined to be the weight of (1). The weight $[\sigma_\mu, k_\mu]$ is said to be *greater* than the weight $[\sigma_\nu, k_\nu]$ if $\sigma_\mu > \sigma_\nu$, or when $\sigma_\mu = \sigma_\nu$, if $k_\mu > k_\nu$. The weight of the complete singularity is defined to be that of the strongest element. When there is only one such element, it is said to be *dominant*, and it is the case of an algebraic-logarithmic singularity with a single dominant element which arises here.

There is no loss of generality and some convenience in taking $\alpha = 1$. Consider the expansion of the algebraic-logarithmic element at $z = 1$

$$(1-z)^{-s}[\log\{(1-z)^{-1}\}]^k \equiv \sum_0^\infty a'_n z^n. \tag{2}$$

Jungen [(**2**) 268–73] has shown that

$$a'_n = \frac{n^{s-1}}{\Gamma(s)} \sum_0^k \phi_\nu(n)(\log n)^{k-\nu} \quad (s \neq 0, -1, -2,\ldots), \tag{3}$$

$$a'_n = (-1)^s k\Gamma(1-s)n^{s-1} \sum_0^{k-1} \phi_\nu(n)(\log n)^{k-\nu-1} \quad (s = 0, -1, -2,\ldots), \tag{4}$$

where $$\phi_\nu(n) \sim c_{\nu 0}+\frac{c_{\nu 1}}{n}+\frac{c_{\nu 2}}{n^2}+\dots.$$

Now consider the algebraic-logarithmic singularity at $z = 1$ with $p$ elements of which only one is dominant,

$$\sum_1^p A_\nu(1-z)^{-s_\nu}[\log\{(1-z)^{-1}\}]^{k_\nu} \equiv \sum_0^\infty a_n z^n. \tag{5}$$

Here, from (2), (3), (5),

$$a_n = \sum_1^p \frac{A_\mu n^{s_\mu-1}}{\Gamma(s_\mu)} \sum_0^{k_\mu} \phi_{\mu\nu}(n)(\log n)^{k_\mu-\nu} \tag{6}$$

with appropriate modifications should the case under (4) arise.

It is convenient to take as an interpolating function for the coefficients $a_n$ of (6) the function $F(z)$ defined below

$$F(z) = \sum_1^p \frac{A_\mu z^{s_\mu-1}}{\Gamma(s_\mu)} \sum_0^{k_\mu} \phi_{\mu\nu}(z)(\log z)^{k_\mu-\nu}, \tag{7}$$

for which $a_n = F(n)$ when $n = 0, 1, 2,\dots$. Here it is of importance to determine the region of validity of the asymptotic expansion of the functions $\phi_{\mu\nu}(z)$. As in (3) and (4), these come from the asymptotic expansions of the $\Gamma$-function and its derivatives. Whittaker and Watson [(**5**) 276–9] give an asymptotic expansion for $\log\Gamma(z)$ valid in $|\arg z| \leqslant \pi-\delta$, for arbitrarily small positive $\delta$, and for every $\delta' > \delta$ this is also valid over an infinite strip about each radius in the region. Hence Lemmas 2 and 3 of Jungen [(**2**) 270] apply, and it follows that the asymptotic expansions of the $\phi_{\mu\nu}(z)$ in (7) are valid in $|\arg z| \leqslant \pi-\delta'$, where $\delta'$ is positive but may be arbitrarily small.

Let the singularity defined by (5) be of weight $[\sigma, k]$. Since there is only one element of greatest weight, it follows from (7) that

$$\lim_{r\to\infty} \frac{|F(r)|}{|r^{\sigma-1}(\log r)^k|} = C, \tag{8}$$

where $C$ is some positive constant and $\operatorname{re} s = \sigma$. Thus, for $r > r_0$, it follows that

$$|F(r)| < e^{\epsilon r}, \qquad |F(r)| > e^{-\epsilon r}, \tag{9}$$

where $r_0$ is determined by the arbitrary positive $\epsilon$. Since $F(z)$ is regular except at the origin and infinity, it follows that both $F(z)$ and $1/F(z)$ satisfy the requirements of the special case of the Le Roy–Lindelöf theorem stated in the Lemma below [(**1**) 340–6], provided that certain adjustments are made.

LEMMA. *Suppose that $G(z)$ is regular in the half-plane* re $z \geqslant \alpha$ *and that, for every arbitrarily small positive $\epsilon$ and for sufficiently large $\rho$,*

$$|G(\alpha+\rho e^{i\psi})| < e^{\epsilon\rho}$$

*for $\frac{1}{2}\pi \leqslant \psi \leqslant \frac{1}{2}\pi$. Then the function represented by $\sum_0^\infty G(n)z^n$ is regular everywhere except perhaps at points on the closed segment $(1,\infty)$ of the positive real axis.*

It is clear from the lemma that coefficients $F(n)$ cannot be interpolated unless $n \geqslant \alpha$. In the same way, when the function $\sum_0^\infty z^n/F(n)$ is under consideration, the coefficients are not interpolated unless $n \geqslant r_0$ to avoid the difficulty arising in the exceptional case in which $F(z)$ may be zero for some $n < r_0$. Without in any way affecting the results of Hadamard multiplication, the initial terms in either of the series $\sum_0^\infty F(n)z^n$, $\sum_0^\infty z^n/F(n)$ can, in fact, be taken arbitrarily. Assuming this to have been done where necessary, it follows that both functions represented by the series $\sum_0^\infty F(n)z^n$ and $\sum_0^\infty z^n/F(n)$ are each regular along the entire plane cut along $(1,\infty)$.

We now consider the Hadamard product of (5) and $g(z) = \sum_0^\infty b_n z^n$, namely $\sum_0^\infty F(n)b_n z^n$. From Theorem A the star-domain of this function contains the product of the star-domains of $\sum_0^\infty F(n)z^n$ and $\sum_0^\infty b_n z^n$. We next consider the Hadamard product

$$\sum_0^\infty F(n)b_n \frac{z^n}{F(n)} = \sum_0^\infty b_n z^n.$$

Again, from Theorem A, it follows that the star-domain of $\sum_0^\infty b_n z^n$ contains the product of the star-domains of $\sum_0^\infty F(n)b_n z^n$ and $\sum_0^\infty z^n/F(n)$. These two results cannot, however, both be true unless the vertices of the star-domains of $\sum_0^\infty b_n z^n$ and of $\sum_0^\infty F(n)b_n z^n$ are the same. In particular, the singularities of each of these functions on their circle of convergence must be at the same points. Hence we have the following theorem:

THEOREM 1. *If the sole singularity of $f(z)$ in the finite part of the plane is an algebraic-logarithmic point with a single dominant element at $z = \alpha$, then*

*the point $\alpha\beta$ is singular for $h(z)$ whatever kind of singularity $g(z)$ may have at the point $z = \beta$ on its circle of convergence.*

By use of a certain process of dissection, this result can be extended to the case in which $\alpha$ and $\beta$ are any vertices of the star-domains of $f(z)$ and $g(z)$ respectively, under conditions stated by Pólya [(3) 768–72 Th. IX], namely: (i) in each case the set of vertices is countable and has no limiting point in the finite part of the plane; (ii) no pairs of indices $(\mu, \nu)$, $(\mu', \nu')$ exist for which $\alpha_\mu \beta_\nu = \alpha_{\mu'} \beta_{\nu'}$. The application of Pólya's methods leads to the theorem below.

THEOREM 2. *If the star-domains of $f(z)$ and $g(z)$ satisfy conditions* (i) *and* (ii) *and if $f(z)$ has an algebraic-logarithmic point with a single dominant element at $z = \alpha$, then the point $\alpha\beta$ is singular for $h(z)$ whatever kind of singularity $g(z)$ may have at a vertex $z = \beta$ of its star-domain.*

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# RINGS OF INFINITE MATRICES

*By* H. F. GREEN (*London*)

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LET $\Sigma(\alpha)$ denote the set of all infinite matrices which map a sequence space $\alpha$ into itself, the series which arise in the transformations being absolutely convergent [298]. It is known that $\Sigma(\alpha)$ is a ring when $\alpha$ contains $\phi$ and is normal [313], whereas $\Sigma(C)$ is not closed under multiplication and is therefore not a ring [307]. The space $C$ is the space of all stationary sequences.

THEOREM. *If $\alpha \supseteq \phi$ and $\Sigma(\alpha)$ is closed under multiplication, then $\Sigma(\alpha)$ is a ring.*

It is sufficient to show that $\Sigma(\alpha) \subseteq \Sigma(\alpha^{**})$ [312]. Suppose that $A \in \Sigma(\alpha)$ and $u \in \alpha^*$. The matrix $U$ whose first row vector is $u$ and whose other row vectors are zero belongs to $\Sigma(\alpha)$, and hence $UA$ belongs to $\Sigma(\alpha)$. The first row vector of the matrix $UA$ is $A'u$. It follows that $A'u \in \alpha^*$ [299]. Hence $A' \in \Sigma(\alpha^*)$ and $A \in \Sigma(\alpha^{**})$ [300]. This proves the theorem.

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# ON DOUBLY STOCHASTIC TRANSFORMS OF A VECTOR

By M. MARCUS (*Vancouver*)

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## Introduction and notation

LET $\lambda = (\lambda_1,..., \lambda_n)$ be an $n$-vector with real coordinates, and consider the $n!$ points $P\lambda$ where $P$ varies over all $n$-square permutation matrices. The convex hull of these points is a set that arises in various contexts (**1**), (**2**), (**5**), (**6**). We investigate here some of its properties when $\lambda$ is a real $n$-vector with *distinct* components. Fortunately there are well-known inequalities [(**1**), (**2**)] describing the set, and our arguments rest heavily on these. In particular we obtain some information about the structure of points lying on support planes of the set.

$S$ is a $p$-square *doubly stochastic* (d.s.) matrix if

$$s_{ij} \geqslant 0, \qquad \sum_{j=1}^{p} s_{ij} = 1, \qquad \sum_{i=1}^{p} s_{ij} = 1$$

for all $i$, $j = 1,..., p$. $\Omega_p$ will denote the totality of d.s. matrices of dimension $p$. If $A$ and $B$ are respectively $p$-square and $q$-square matrices, then $A \dot{+} B$ is their $(p+q)$-square direct sum. More generally, if $A_j$ is a $p_j \times q_j$ matrix for $j = 1,..., m$, then $\sum_{j=1}^{m} . A_j$ denotes the direct sum. If $A$ is any matrix, it will prove convenient to distinguish the sums of the elements by rows and by columns as follows:

$$\sigma_r(A) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}, \qquad \sigma_c(A) = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}.$$

A *partition* of a set $\alpha$ of $m$ integers ordered in increasing size will be a sequence $[k_1, k_2,..., k_t]$ of positive integers such that $\alpha$ can be decomposed into $t$ disjoint sets $\alpha_1,..., \alpha_t$ with $k_i$ elements in $\alpha_i$ satisfying the conditions:

(*a*) the elements of each $\alpha_j$ are arranged in increasing size;

(*b*) if $x \in \alpha_j$ and $y \in \alpha_s$ and $j < s$, then $x < y$;

(*c*) $\sum_{j=1}^{t} k_j = m$.

If $\pi_1$ and $\pi_2$ are any two partitions of $\alpha$, then $\pi_1 \pi_2$ denotes their common refinement. For example, if $\alpha = \{1, 2,..., 9\}$ and

$$\pi_1 = [2, 3, 4], \qquad \pi_2 = [5, 3, 1],$$

then $\pi_1\pi_2 = [2,3,3,1]$. It is obvious that $\pi_1\pi_2$ has at least as many elements as $\pi_1$ or $\pi_2$ and at least one more if $\pi_1 \neq \pi_2$.

We assume, unless it is otherwise stated, that $\lambda_j > \lambda_{j+1}$ $(j = 1,\dots, n-1)$ and we denote by $H_n(\lambda)$ the convex hull of the points $P\lambda$ where $P$ varies over all the $n!$ permutation matrices in $\Omega_n$. It is well known (4) that $H_n(\lambda)$ is the intersection of the following half-spaces and hyperplane:

$$\sum_{j=1}^{k} t_{i_j} \leqslant \sum_{j=1}^{k} \lambda_j, \tag{1}$$

$$\sum_{j=1}^{n} t_j = \sum_{j=1}^{n} \lambda_j, \tag{2}$$

where $1 \leqslant k < n$, $1 \leqslant i_1 < \dots < i_k \leqslant n$ and the $t_r$ are current coordinates. Moreover, from Birkhoff's result on $\Omega_n$ (2), we know that

$$H_n(\lambda) = \{t \mid t = S\lambda,\ S \in \Omega_n\}. \tag{3}$$

A complete account of the connexions between $H_n(\lambda)$ and $\Omega_n$ is contained in (5) Theorem 1.

## Results

(The author wishes to express his thanks to Dr. Morris Newman and Professor Ky Fan for their valuable suggestions.)

THEOREM 1. *Let $\pi_1 = [k_1,\dots, k_m]$, $\pi_2 = [e_1,\dots, e_r]$ be partitions of the integers $1,\dots, n$ and assume that*

$$S = P\sum_{j=1}^{m}{}^{\boldsymbol{\cdot}} S_j = \sum_{j=1}^{r}{}^{\boldsymbol{\cdot}} T_j \tag{4}$$

*with $S_j \in \Omega_{k_j}$, $T_j \in \Omega_{e_j}$ and $P$ a permutation matrix in $\Omega_n$. Then, if $\pi = \pi_1\pi_2 = [p_1,\dots, p_v]$, there exist $K_j \in \Omega_{p_j}$ $(j = 1,\dots, v)$ and a permutation matrix $R \in \Omega_n$ such that*

$$RS = \sum_{j=1}^{v}{}^{\boldsymbol{\cdot}} K_j. \tag{5}$$

*Proof.* The proof is by induction on $n$ with nothing to prove for $n = 1$. We suppose as the first case that $k_1 = e_1$. Then write (4) as

$$P\Big(S_1 \dotplus \sum_{j=2}^{m}{}^{\boldsymbol{\cdot}} S_j\Big) = T_1 \dotplus \sum_{j=2}^{r}{}^{\boldsymbol{\cdot}} T_j. \tag{6}$$

We first assert that the permutation $\tau_p$ corresponding to $P$ has the property

$$\tau_p(j) \in \{1,\dots, k_1\} \tag{7}$$

for $j \in \{1,\dots, k_1\}$. For, if $\tau_p(j) > k_1$, there would be at least one non-zero element somewhere in row $\tau_p(j)$ and in one of the columns $1,\dots, k_1$ of $\sum_{j=1}^{r}{}^{\boldsymbol{\cdot}} T_j$, which is a contradiction.

Thus by (7) we have $$P = P_1 \dot{+} P_2 \tag{8}$$
with $\tau_{p_2}(j) \in \{k_1+1,\dots,n\}$ for $j \in \{k_1+1,\dots,n\}$. Define $L \in \Omega_{n-k_1}$ by
$$L = P_2 \sum_{j=2}^{m} .S_j = \sum_{j=2}^{r} .T_j \tag{9}$$
and apply the induction hypothesis to obtain a permutation matrix $R_1 \in \Omega_{n-k_1}$ and a set of $E_j \in \Omega_{u_j}$, where
$$\pi_3 = [k_2,\dots,k_m][e_2,\dots,e_r] = [u_1,\dots,u_c],$$
such that $$R_1 L = \sum_{j=1}^{c} .E_j. \tag{10}$$
But then
$$S = P\Big(S_1 \dot{+} \sum_{j=2}^{m} .S_j\Big) = P_1 S_1 \dot{+} P_2 \sum_{j=2}^{m} .S_j = P_1 S_1 \dot{+} L = P_1 S_1 \dot{+} R_1' R_1 L$$
$$= P_1 S_1 \dot{+} R_1' \sum_{j=1}^{c} .E_j = (P_1 \dot{+} R_1')\Big(S_1 \dot{+} \sum_{j=1}^{c} .E_j\Big). \tag{11}$$
It is clear that
$$[k_1, u_1,\dots,u_c] = [k_1, k_2,\dots,k_m][e_1, e_2,\dots,e_r],$$
and we achieve the statement (5) by setting
$$R = P_1 \dot{+} R_1', \qquad K_1 = S_1, \qquad K_{j+1} = E_j, \qquad p_1 = k_1,$$
$$p_{j+1} = u_j, \qquad v = c+1$$
for $j = 1,\dots, c$.

We next consider the case $k_1 \neq e_1$ and we assume without loss of generality that $k_1 > e_1$. Now, since there exist $n-e_1$ rows of $\sum_{j=1}^{r} .T_j$ with zeros in the first $e_1$ columns, it follows that
$$k_1 - e_1 = (n-e_1)-(n-k_1)$$
rows of $S_1$ must have zeros in the first $e_1$ columns.

It is then clear that we can choose a permutation matrix $Q \in \Omega_n$ such that
$$Q \sum_{j=1}^{m} .S_j = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \dot{+} \sum_{j=2}^{m} .S_j \tag{12}$$
with dimensions as follows:
$$S_{11}\colon\ e_1 \times e_1, \qquad S_{21}\colon\ (k_1-e_1)\times e_1,$$
$$S_{12}\colon\ e_1 \times (k_1-e_1), \qquad S_{22}\colon\ (k_1-e_1)\times(k_1-e_1),$$
and $S_{21}$ a matrix of zeros. We have
$$\sigma_c(S_{11}) = e_1 = \sigma_r(S_{11}), \qquad \sigma_r(S_{12}) = 0,$$

and thus $S_{12} = 0$. Hence, from (12),

$$S = P\sum_{j=1}^{m}{}^{\cdot} S_j = PQ'Q\sum_{j=1}^{m}{}^{\cdot} S_j = PQ'\Big(S_{11} \dot{+} S_{22} \dot{+} \sum_{j=2}^{m}{}^{\cdot} S_j\Big) = \sum_{j=1}^{r}{}^{\cdot} T_j.$$

But now $S_{11}$ and $T_1$ are the same size, which is the situation in the first case, and the proof is complete.

We see from (3) that any point $t \in H_n(\lambda)$ is of the form $t = S\lambda$ for $S \in \Omega_n$. The next result describes the structure of $S$ when $t$ lies on *one* of the support planes (1).

THEOREM 2. *If $t = S\lambda$ lies on one of the support planes* (1) *of $H_n(\lambda)$, there exist positive integers $p$ and $q$, a pair of d.s. matrices $S_1$, $S_2$*

$$(S_1 \in \Omega_p,\ S_2 \in \Omega_q)$$

*and a permutation matrix $P \in \Omega_n$ such that $p+q = n$ and*

$$t = P(S_1 \dot{+} S_2)\lambda.$$

*Proof.* We remark first that for purposes of this argument it is only necessary to have $\lambda_j \geqslant \lambda_{j+1}$ for $j = 1,..., n-1$.

Suppose that $t$ satisfies

$$t_{i_1}+t_{i_2}+...+t_{i_p} = \lambda_1+...+\lambda_p, \tag{13}$$

where $1 \leqslant p \leqslant n-1$. It follows from (2) that, if $j_1,..., j_q$ is the complementary set of $i_1,..., i_p$ in $1,..., n$, then

$$t_{j_1}+...+t_{j_q} = \lambda_{p+1}+...+\lambda_n. \tag{14}$$

Also, if $e_1,..., e_r$ is a subset of $j_1,..., j_q$, then

$$t_{e_1}+...+t_{e_r}+t_{i_1}+...+t_{i_p} \leqslant \lambda_1+...+\lambda_{r+p},$$

i.e. by (13),

$$t_{e_1}+...+t_{e_r} \leqslant \lambda_1+...+\lambda_{r+p}-(\lambda_1+...+\lambda_p) = \lambda_{p+1}+...+\lambda_{r+p}. \tag{15}$$

Now set

$$t^p = (t_{i_1},..., t_{i_p}), \qquad t^q = (t_{j_1},..., t_{j_q}), \qquad \lambda^p = (\lambda_1,..., \lambda_p),$$
$$\lambda^q = (\lambda_{p+1},..., \lambda_n)$$

and from (13), (14), (15), (1), and (3) we conclude that

$$t^p = S_1\lambda^p, \qquad t^q = S_2\lambda^q$$

with $S_1 \in \Omega_p$, $S_2 \in \Omega_q$. Choose a permutation matrix $P \in \Omega_n$ such that

$$P(t^p \dot{+} t^q) = t,$$

and then

$$t = P(t^p \dot{+} t^q) = P(S_1\lambda^p \dot{+} S_2\lambda^q) = P(S_1 \dot{+} S_2)(\lambda^p \dot{+} \lambda^q) = P(S_1 \dot{+} S_2)\lambda.$$

We next consider the extent to which this representation is unique.

THEOREM 3. *Let $t = (S_1 \dotplus S_2)\lambda$ ($S_1 \in \Omega_p$, $S_2 \in \Omega_q$, $p+q = n$) and suppose that $t = T\lambda$, $T \in \Omega_n$. Then there exist $T_1 \in \Omega_p$ and $T_2 \in \Omega_q$ such that*

$$T = T_1 \dotplus T_2. \tag{16}$$

*Proof.* Let $S_1 = (a_{ij})$, $S_2 = (b_{ij})$, $T = (d_{ij})$. Then

$$t = T\lambda = (S_1 \dotplus S_2)\lambda$$

implies
$$\sum_{j=1}^{p} a_{ij}\lambda_j = \sum_{j=1}^{n} d_{ij}\lambda_j \quad (i = 1,..., p). \tag{17}$$

Put
$$s_j = \sum_{i=1}^{p} d_{ij} \quad (0 \leqslant s_j \leqslant 1;\ j = 1,..., n).$$

Then
$$\sum_{r=1}^{n} s_r = \sum_{i=1}^{p} \sum_{j=1}^{n} d_{ij} = p. \tag{18}$$

Now sum the equations (17) to obtain

$$\sum_{j=1}^{p} \lambda_j = \sum_{j=1}^{p} s_j \lambda_j + \sum_{j=p+1}^{n} s_j \lambda_j. \tag{19}$$

Also note from (18) that

$$\sum_{=1}^{p}(1-s_j) = p - \sum_{j=1}^{p} s_j = p - \Big(\sum_{j=1}^{n} s_j - \sum_{j=p+1}^{n} s_j\Big)$$
$$= p - \Big(p - \sum_{j=p+1}^{n} s_j\Big) = \sum_{j=p+1}^{n} s_j,$$

and
$$\sum_{j=p+1}^{n} s_j \lambda_j \leqslant \lambda_{p+1} \sum_{j=p+1}^{n} s_j = \lambda_{p+1} \sum_{j=1}^{p}(1-s_j).$$

Combining this with (19) we have

$$\sum_{j=1}^{p}(1-s_j)\lambda_j \leqslant \lambda_{p+1} \sum_{j=1}^{p}(1-s_j), \qquad \sum_{j=1}^{p}(1-s_j)(\lambda_j - \lambda_{p+1}) \leqslant 0. \tag{20}$$

Since $\lambda_j - \lambda_{p+1} > 0$ for $j = 1,..., p$, we conclude from (20) that $s_j = 1$ ($j = 1,..., p$). Now let

$$T = \begin{pmatrix} T_1 & T_4 \\ T_3 & T_2 \end{pmatrix}$$

with dimensions

$$T_1\colon\ p \times p, \qquad T_4\colon\ p \times (n-p) = p \times q,$$
$$T_3\colon\ (n-p) \times p = q \times p, \qquad T_2\colon\ (n-p) \times (n-p) = q \times q.$$

Then
$$\sigma_c(T_1) + \sigma_c(T_3) = p = \sum_{j=1}^{p} s_j = \sigma_c(T_1);$$

so
$$\sigma_c(T_3) = 0, \qquad T_3 = 0.$$

Also
$$\sigma_r(T_1) = \sigma_c(T_1) = p, \qquad \sigma_r(T_2) = q,$$

and hence

$$n = \sigma_r(T) = \sum_{j=1}^{4} \sigma_r(T_j) = p+\sigma_r(T_4)+n-p = n+\sigma_r(T_4).$$

Thus

$$T_4 = 0.$$

This completes the proof.

Combining our results so far we have

THEOREM 4. *If $t = S\lambda$ is on a support plane* (1), *then*

$$S = P(S_1 \dotplus S_2) \tag{21}$$

*with the same notation as Theorem* 2.

We are now in a position to discuss the structure of $S$ when $t = S\lambda$ lies on more than one of the support planes (1). We shall say that point $t \in H_n(\lambda)$ *lies on $k$ support planes of different size of* $H_n(\lambda)$ if there exist a set of positive integers $1 \leqslant m_1 < m_2 < \dots < m_k \leqslant n-1$ and a collection of sequences of positive integers

$$1 \leqslant i_{j1} < i_{j2} < \dots < i_{jm_j} \leqslant n \quad (j = 1,\dots, k)$$

such that

$$\sum_{g=1}^{m_j} t_{i_{jg}} = \sum_{g=1}^{m_j} \lambda_g \quad (j = 1,\dots, k).$$

THEOREM 5. *If $t = S\lambda$ lies on $k$ support planes* (1) *of different size of $H_n(\lambda)$, then there exist a permutation matrix $R \in \Omega_n$ and a partition $[u_1,\dots, u_m]$ of $1,\dots, n$ such that $m \geqslant k+1$ and*

$$S = R\sum_{j=1}^{m} {}^{\cdot} K_j \tag{22}$$

*with $K_j \in \Omega_{u_j}$ $(j = 1,\dots, m)$.*

*Proof.* We see from Theorem 4 that for each $m_j$ there exist $S_{1j} \in \Omega_{m_j}$, $S_{2j} \in \Omega_{n-m_j}$ and a permutation matrix $P_j \in \Omega_n$ for which

$$S = P_j(S_{1j} \dotplus S_{2j}) \quad (j = 1,\dots, k). \tag{23}$$

Then

$$P_2' P_1(S_{11} \dotplus S_{21}) = (S_{12} \dotplus S_{22}).$$

It follows from Theorem 1 (since $m_1 < m_2$) that we can secure a permutation matrix $R \in \Omega_n$ such that

$$RS = T_1 \dotplus T_2 \dotplus T_3.$$

We repeat this process using $RS$ and $P_3(S_{13} \dotplus S_{23})$ and so on until the relations (23) are exhausted. The fact that the $m_j$ are increasing assures us that at each stage we introduce at least one new block in the decomposition of $S$, and hence there are at least $k+1$ blocks at the end of the process. This completes the proof.

*Some Remarks.* It is clear from (22) that, if $k = n-1$, then $t$ is a vertex of $H_n(\lambda)$. If $k = n-2$, then

$$S = \theta P + (1-\theta)Q \quad (0 \leqslant \theta \leqslant 1)$$

for $P$ and $Q$ permutation matrices, and $t$ lies on an edge.

Now suppose that $\lambda$ does not have distinct coordinates: say

$$\lambda = (\lambda_1(k_1 \text{ times}),..., \lambda_m(k_m \text{ times}))$$

$(\lambda_1 > \lambda_2 > ... > \lambda_m)$. It is not difficult to show that, if $Q\lambda = S\lambda$ for $Q$ a permutation matrix, then there is a permutation matrix $P$ such that

$$P'Q'SP = \sum_{j=1}^{m} \cdot S_j \quad (S_j \in \Omega_{k_j};\ j = 1,..., m). \tag{24}$$

The relation (24) has as a corollary the fact that the number of distinct coordinates of an eigenvector corresponding to the eigenvalue 1 of a d.s. matrix $S$ is at least the algebraic multiplicity of 1. This can also be derived from a result of A. Brauer (**3**).

There are several questions that arise concerning the set $H_n(\lambda)$. Suppose that $\lambda$ is a complex $n$-vector, $\lambda = \mu + i\omega$. If the $\mu_i$ and $\omega_i$ are distinct, can Theorem 2 be extended, i.e. is it true that, if $t \in \partial H_n(\lambda)$, then

$$t = (S_1 \dot{+} S_2)Q\lambda,$$

where $P$ and $Q$ are permutation matrices, $\partial H_n(\lambda)$ being the topological boundary of $H_n(\lambda)$?

Again assume $\lambda$ real with distinct coordinates. Under what conditions do the points $P_1\lambda,..., P_{m+1}\lambda$ span an $m$-simplex, where the $P_j$ are distinct permutation matrices? This, of course, amounts to examining the rank of the matrix whose columns are $P_1\lambda,..., P_m\lambda$. The answer for $m \leqslant 4$ is known.

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